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A new covariant form of the equations of geophysical fluid dynamics and their structure-preserving discretization

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Set of geophysical fluid equations:

1. Define mathematical descriptors for: (i) particle movements
   (ii) force fields
2. Solve balance of mass, momentum and energy $\Rightarrow$ fluid equations

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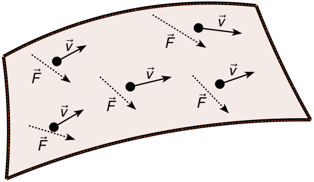
Descriptors mostly used in geophysical fluid dynamics (GFD):

- Particle movements $\leftrightarrow$ Vectors $\vec{v}$
- Forces $\leftrightarrow$ Vectors $\vec{F}$



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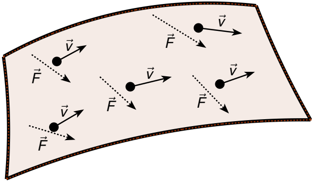

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Discretize vector-valued equations by, e.g. finite difference, finite element, ...

Alternative mathematical descriptors

Are there *other* descriptors to represents the physical entities?

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$\Rightarrow$ *form* of fluid equations: vector-invariant

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- Particle movements $\leftrightarrow$ Vectors $\vec{v}$
- Forces $\leftrightarrow$ ~~Vectors $\vec{F}$~~ Linear map: Vectors $\rightarrow$ Reals

$$\mathcal{F} : \vec{v} \rightarrow \mathcal{F}(\vec{v}) \in \mathbb{R}$$

$\Rightarrow$ *form* of fluid equations: ~~vector-invariant~~ covariant

$$\cancel{\frac{\partial \vec{v}}{\partial t} + g \nabla h = 0} \quad \forall \vec{v} : \frac{\partial}{\partial t} u(\vec{v}) + g \mathbf{d} h(\vec{v}) = 0$$

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Parts of this talk:

1. What are optimal mathematical descriptors?
2. How do covariant equations of GFD look like?
3. Application: derivation of structure-preserving discretization

Outline

Description of movement of fluid particles

Physical properties of particle displacement:

- Displacement has direction and magnitude
- All positions should be accessible

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Mathematical model:

- Affine translation $a_{\vec{v}} = x \to x + \vec{v}$ in affine space A_n
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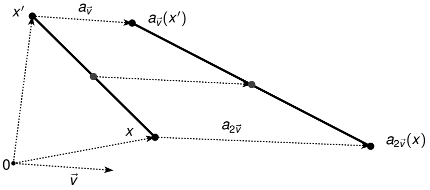
Conserved quantities under $a_{\vec{v}}$:

- Alignment and barycenter of points
- Ratio of distance between aligned pnts
- Parallel lines remain parallel

Not conserved: distances, angles, ...



Description of forces

Physical properties of forces*:

- are *not directly* visible, only by e.g. particle displacement
- cause displacement $\delta\vec{v}$ that corresponds to work $W = F \cdot |\delta\vec{v}|$

*following Bossavit 99

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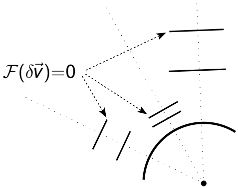
Properties of **affine** covectors:

- $\delta\vec{v}$ in kernel of $\mathcal{F}_x$ span equipotential surfaces
- Line distance encodes amount of virtual work performed



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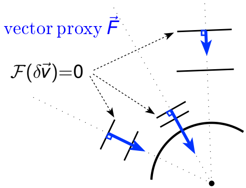

Given a metric: vector proxy $\vec{F}$ given by $\mathcal{F}_x = <\vec{F}, >$;

- $\vec{F}$ metric-dependent: $\mathcal{F}_x = <\vec{F}_{\mathrm{inch}}, >_{\mathrm{inch}} = <\vec{F}_{\mathrm{m}}, >_{\mathrm{m}}$
- $\vec{F}$ describes only particle trajectories, not whole field

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Outline

Topological equations:

$$\partial_t \int_c u + \int_c \mathbf{i}_{\vec{V}}(\mathbf{d}u + 2\Omega_{\mathrm{rot}}) + \int_{\partial c} \left(\frac{p}{\rho} + w + \kappa + \Phi_{A^n} \right) = 0\,, \quad \partial_t \int_V \widetilde{\rho} + \int_{\partial V} \widetilde{\rho u} = 0\,,$$

Metric equations:

$$\widetilde{\star}\rho u = \widetilde{\rho u}\,, \qquad \widetilde{\star}\rho = \widetilde{\rho}\,, \qquad u^\sharp = \vec{V}\,,$$

with energy closure for (i) *incompressible* or (ii) *barotropic* flows.

- n-dimensional rotating fluid equations
- Independent of choice of orientation
- Split equations agree with vector-invariant ones in $\mathbb{R}^3$

*Bauer 2014 (submitted to GEM)

A split covariant form of n-dimensional rotating Euler's equations*:

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Structure of equations suggests how to discretize them

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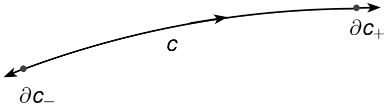
Topological momentum equation

$\partial_t \int_c u =$

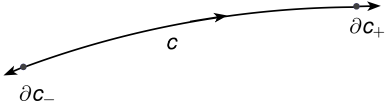


- Inner orientation of c orients boundary
- Terms in units of specific energy density $[J/kg]$

$$\partial_t \int_c u =$$

$u : \mathcal{X}(\mathcal{M}) \to \mathbb{R}$ velocity 1-form $u \in \Omega^1(\mathcal{M})$

$$\partial_t \int_c u = -\int_c \mathbf{i}_{\vec{V}} \zeta_{\mathrm{rel}}$$



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$\zeta_{\mathrm{rel}} : \mathcal{X}(\mathcal{M}) \times \mathcal{X}(\mathcal{M}) \to \mathbb{R}$	rel. vorticity 2-form $\zeta_{\mathrm{rel}} \in \Omega^2$
$\mathbf{d} : \Omega^k(\mathcal{M}) \to \Omega^{k+1}(\mathcal{M})$	exterior derivative
$\mathbf{i}_{\vec{V}} : \Omega^k(\mathcal{M}) \to \Omega^{k-1}(\mathcal{M})$	interior product
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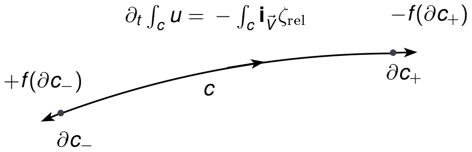

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$$\partial_t \int_c u = -\int_c \mathbf{i}_{\vec{V}}(\overbrace{\mathbf{d}u + 2\Omega_{\mathrm{rot}}}^{\zeta_{\mathrm{rel}}}) - \int_{\partial c}\Big(\overbrace{\frac{p}{\rho} + w + \kappa + \Phi_{A^n}}^{=f}\Big)$$

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$\rho, p, w, \kappa, \Phi_{A^n} \in \Omega^0(\mathcal{M})$	density, pressure, inner and kin. energy, grav. pot.

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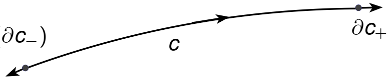

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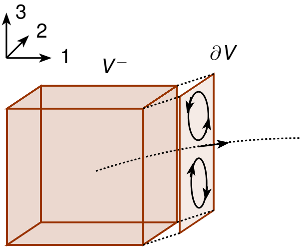
Top. momentum equation is independent of metric and orientation

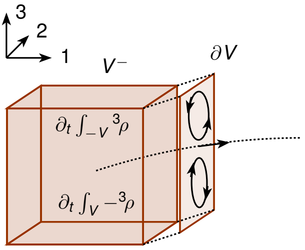
Topological continuity equations



in Or

in -Or

- Outer orientation of V^- and ∂V



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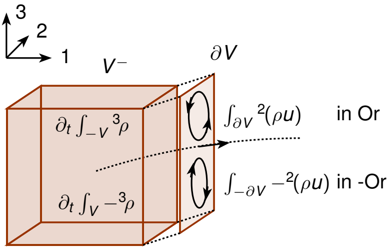
- Outer orientation of V^- and ∂V
- Twisted forms: $\widetilde{\rho} := \{\{{}^n\rho, Or\}, \{-{}^n\rho, -Or\}\}$
- Terms in units of mass-flux $[kg/s]$

$$-\partial_t \int_V \widetilde{\rho} =$$

$\widetilde{\rho} : \mathcal{X}(\mathcal{M})^1 \times ... \times \mathcal{X}(\mathcal{M})^n \to \mathbb{R}$ density n-form $\widetilde{\rho} \in \Omega^n(\mathcal{M})$

Topological continuity equations



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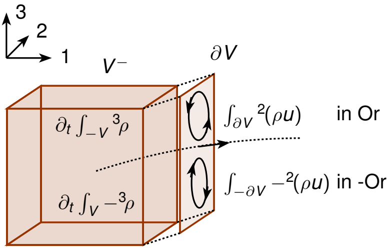

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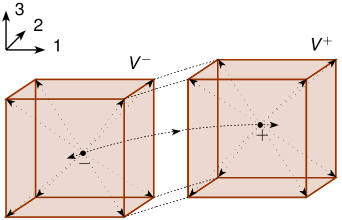
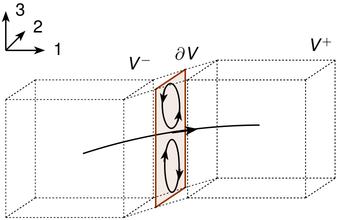
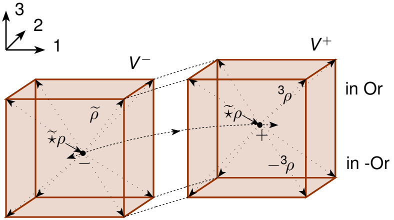
Top. continuity equation is independent of metric and orientation

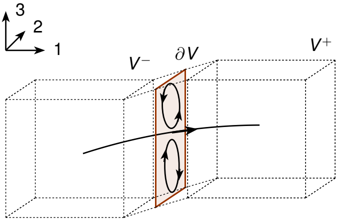
Connection between top. momentum and top. continuity equations





- Matching orientations: $\partial C_{\pm}$ outer orients $V^{\pm}$, c outer orients ∂V





$$\widetilde{\star} : \rho \rightarrow \widetilde{\rho}$$

$$[m^3] : [kg/m^3] \rightarrow [kg]$$

- Matching orientations: $\partial C_\pm$ outer orients $V^\pm$, c outer orients ∂V
- Unique map: $\widetilde{\star} : \rho \rightarrow \widetilde{\rho} := \{\{\star^3\rho, Or\}, \{-\star{}^3\rho, -Or\}\}$
  - $\star : \Omega^k(\mathcal{M}) \rightarrow \Omega^{(n-k)}(\mathcal{M})$ in n dimensions

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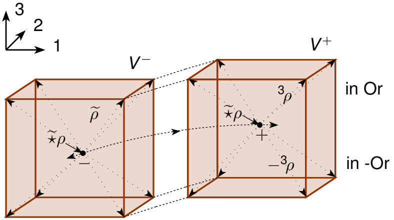

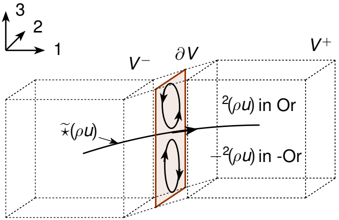

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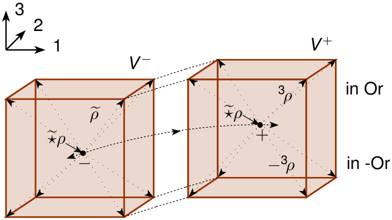

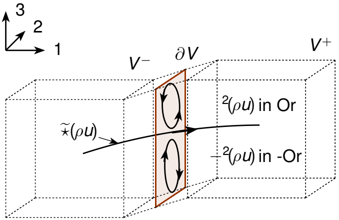

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Metric equations close the set of equations and provide metric information

Outline

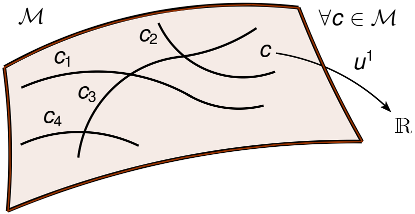
Example: **Split linear shallow-water equations**

Systematic discretization method:

1. Discretize manifolds by computational meshes
2. Discretize differential forms by interpolation functions, e.g. finite element (piecewise constant, piecewise linear, ...)
3. Discretize metric equations, e.g. using diagonal matices

Discretization of the topological *momentum* shallow-water equation:

- Integral form: $\forall c \in \mathcal{M}$:
  $\partial_t \int_c u^1 + g \int_{\partial c} h = 0$
- Approximate manifolds:
  $\mathcal{M} \approx K$, $c \approx \sum_e e$
- Approximate forms:
  u^1, h piecewise constant
  $\Rightarrow u_e \approx \int_e u^1; \pm h_v \approx \int_{\partial e} h$



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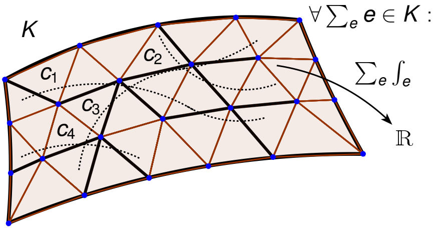

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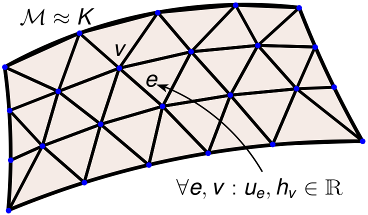


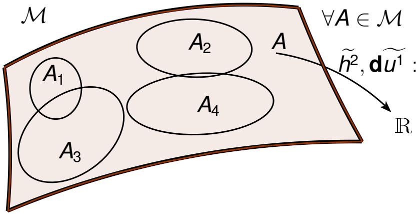
$\Rightarrow$ Matrix form of the **topological** momentum equation:

$$\partial_t \begin{pmatrix} u_{e_1} \\ \vdots \\ u_{e_{|K^e|}} \end{pmatrix} + g \begin{pmatrix} 0 & -1 \dots & 1 \dots & 0 \\ \vdots & \ddots & & \vdots \\ -1 & 0 \dots & 1 \dots & 0 \end{pmatrix}_{\mathbf{G}} \begin{pmatrix} h_{v_1} \\ \vdots \\ h_{v_{|K^v|}} \end{pmatrix} = 0$$

- $\mathbf{G} \in M(|K^e| \times |K^v|)$ is ± 1 if $v \in \partial e$
- Algebraic equation is metric-free

Discretization of the topological *continuity* shallow-water equation:

- Integral form: $\forall A \in \mathcal{M}$:
  $\partial_t \int_A \widetilde{h^2} + H \int_{\partial A} \widetilde{u^1} = 0$
- Approximate manifolds:
  $\mathcal{M} \approx K, A \approx \sum_f f$
- Approximate forms:
  $\widetilde{h^2}, \widetilde{u^1}$ piecewise constant
  $\widetilde{h}_f \approx \int_f \widetilde{h^2}, \pm\widetilde{u}_{\partial f} \approx \int_{\partial f} \widetilde{u^1}$



Discretization of the topological *continuity* shallow-water equation:

- Integral form: $\forall A \in \mathcal{M}$:
  $\partial_t \int_A \widetilde{h^2} + H \int_{\partial A} \widetilde{u^1} = 0$
- Approximate manifolds:
  $\mathcal{M} \approx K$, $A \approx \sum_f f$
- Approximate forms:
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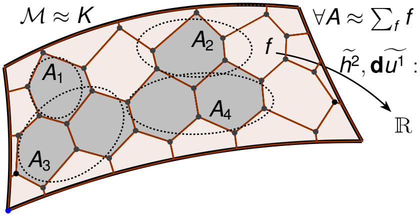
  $\widetilde{h}_f \approx \int_f \widetilde{h^2}, \pm\widetilde{u}_{\partial f} \approx \int_{\partial f} \widetilde{u^1}$



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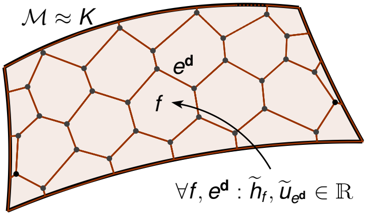

$\Rightarrow$ Matrix form of the **topological** continuity equation:

$$\partial_t \begin{pmatrix} \widetilde{h}_{f_1} \\ \vdots \\ \widetilde{h}_{f_{|K^f|}} \end{pmatrix} + H \begin{pmatrix} 0 & -1 \ldots & 1 \ldots & 0 \\ \vdots & \ddots & & \vdots \\ -1 & 0 \ldots & 1 \ldots & 0 \end{pmatrix}_{\mathbf{D}^{\mathrm{d}}} \begin{pmatrix} \widetilde{u}_{e_1^{\mathrm{d}}} \\ \vdots \\ \widetilde{u}_{e^{\mathrm{d}}_{|K^e|}} \end{pmatrix} = 0$$

- $\mathbf{D}^{\mathrm{d}} \in M(|K^f| \times |K^e|)$ is ± 1 if $e^{\mathrm{d}} \in \partial f$
- Algebraic equation is metric-free

Discretization of *metric* shallow-water equations

- Continuous metric equations:
  $\widetilde{\star} : h \to \widetilde{h^2}, \ \widetilde{\star} : u^1 \to \widetilde{u^1}$
- Approximate $\widetilde{\star}$ by **diagonal** matrices:
  - $\star_0 : \mathbf{h} \to \widetilde{\mathbf{h}} = \frac{A_f}{1} \mathbf{h}$
  - $\star_1 : \mathbf{u} \to \widetilde{\mathbf{u}} = \frac{d_e}{l_e} \mathbf{u}$

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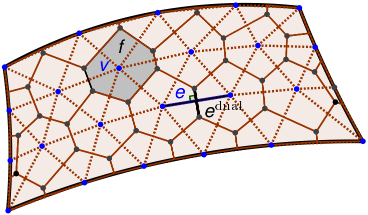

- $\star_0, \star_1$ connect algebraic momentum and continuity equation
- Endows metric-free equation with metric information

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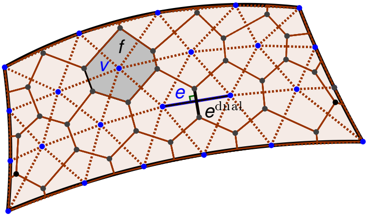

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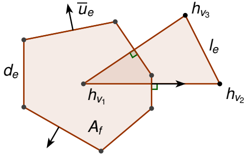
- Endows metric-free equation with metric information

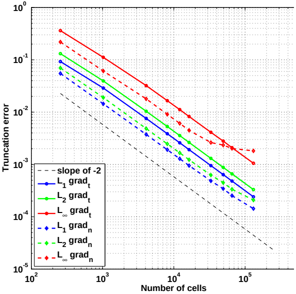
Resulting scheme $\to$ hexagonal C-grid

$$\partial_t \overline{u}_e + g \underbrace{\frac{h_{v_2} - h_{v_1}}{l_e}}_{=\mathrm{grad}_t(h_v)} = 0\,, \quad \partial_t h_v + H \cdot \underbrace{\frac{1}{A_f} \sum_{i=1}^{\#\mathrm{edges}} (\pm) d_{e_i} \overline{u}_{e_i}}_{=\mathrm{div}_v(\overline{u}_e)} = 0\,,$$

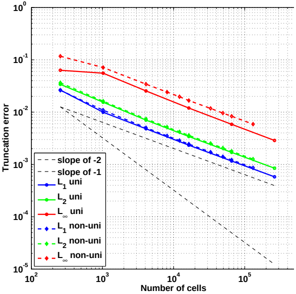


- Switch primal and dual mesh $\Rightarrow$ triangular C-grid scheme
- Tri/hex are structure-presering: $\mathbf{R} \cdot \mathbf{G} = 0$ and $\mathbf{D} \cdot \mathbf{R} = 0$
- Schemes are stable on uniform/non-uniform meshes

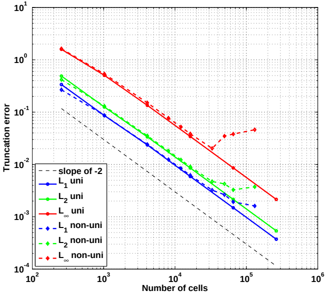
$\mathrm{grad}_t, \mathrm{grad}_n$ (non-uniform)



operators on triangles (curl_{cc})



operators on hexagons (div_v)



Outline

Summary

- Differential forms optimally describe force fields
- Using straight/twisted differential forms, the equations are independent of orientation
- Split equations are ordered with respect to mathematical structures required (affine space + metric + orientation)
- Similarly structured discrete equations exist in literature (Cotter and Thuburn 2013, Bossavit 2005)
- The split form proposes a systematic discretization using algebraic approaches or finite element exterior calculus

Outlook

- Further analytical studies of the split equations
- Use higher order FE to approximate differential forms
- Use non-diagonal Hodge star matrices