Non-hydrostatic sound-proof equations of motion for gravity-dominated compressible flows

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I. Approximate equations of motion
- Hamilton's principle and asymptotics
- Soundproofing : anelastic vs hydrostatic

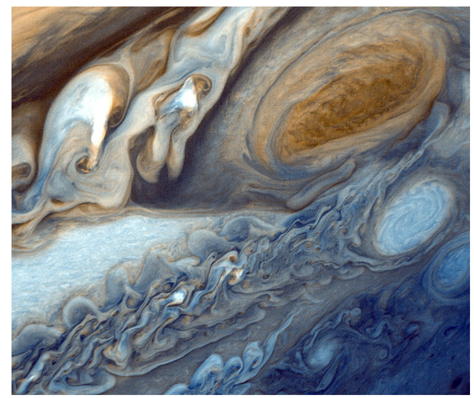
II.A « semi-hydrostatic » approximation
- Derivation and interpretation
- « Elliptic » problem for non-hydrostatic pressure
- Accuracy : normal-mode analysis

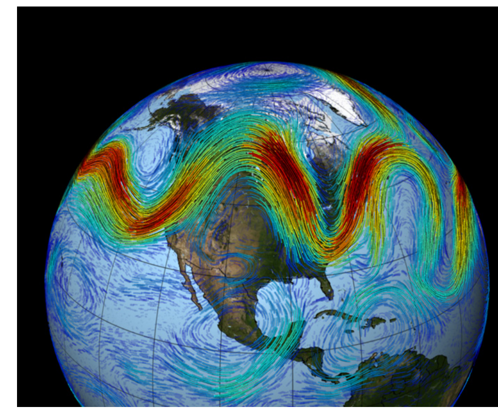
III.Conclusions

The atmosphere : a gravity-dominated, compressible flow

Characteristic scales

- *Velocity* : Sound $c \sim 340$m/s — Wind $U \sim 30$m/s
- *Time* : Buoyancy oscillations $N \sim g/c \sim 10^{-2}\,s^{-1}$ — Coriolis $f \sim 10^{-4}\,s^{-1}$
- *Length* : **Scale height $H=c^2/g=10$km** — **Rossby radius : $R=c/f \sim 1000$ km**





small-scale — mesoscale — synoptic — planetary

1 km — **10 km** — 100 km — **1000 km** — 10000 km

- *Mach number : $M=U/c \ll 1$* — *Scale separation : $f/N \sim H/R \ll 1$*
- *Small numbers => asymptotics => approximate equations*
- ***But not at the expense of conservation : energy, momentum, potential vorticity***

Can we make approximations

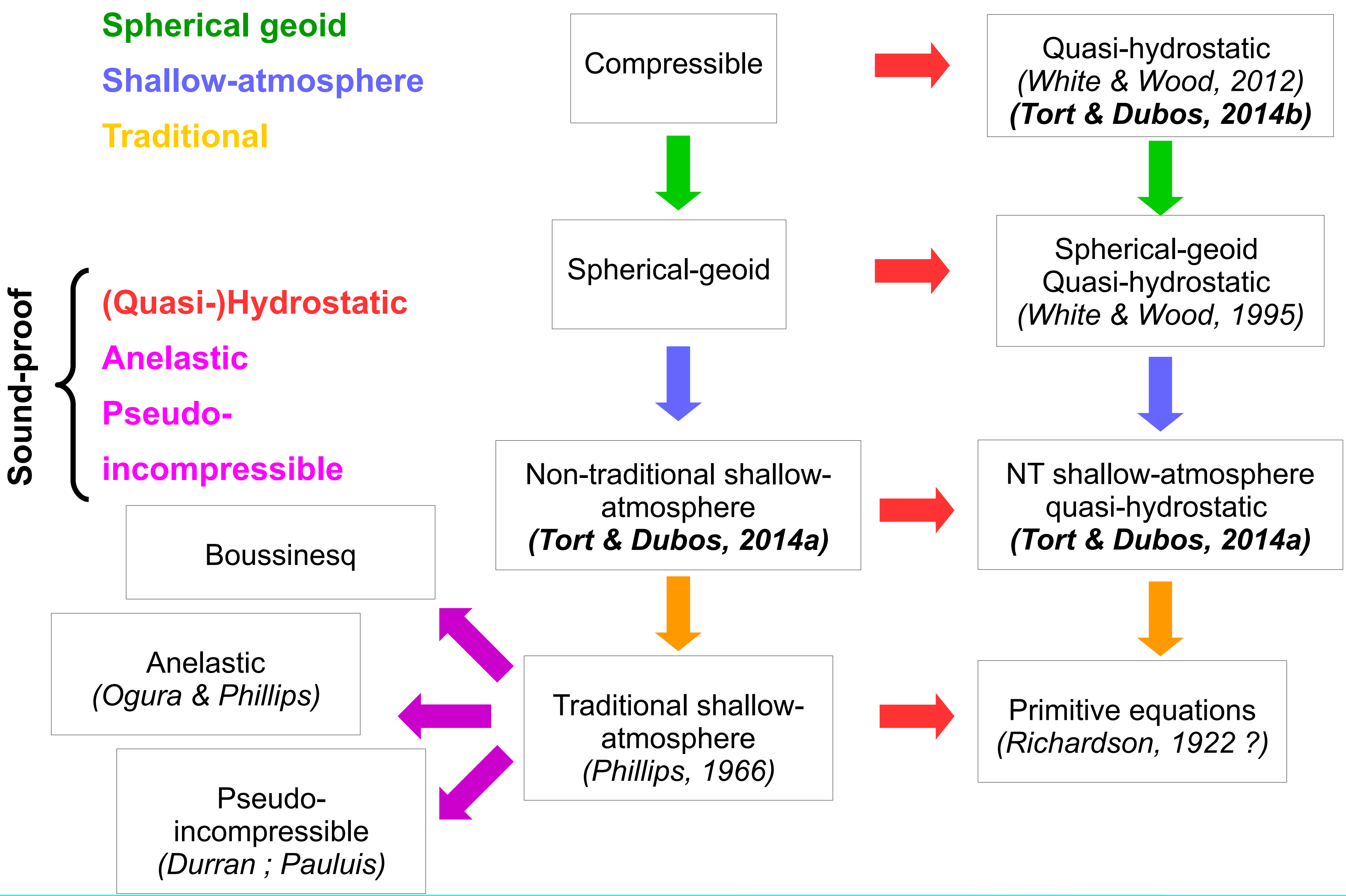
while always maintaining conservation laws ?

- Maupertuis, Euler, Lagrange
- 1834 Hamilton — *Hamilton's least action principle (HP)*
- 1915 Noether — *Noether's theorems*
- 1960 Eckart — *HP for ideal fluid (Lagrangian description)*
- 1962 Newcomb — *HP for ideal fluid (Eulerian description)*
- 1996 Padhye & Morrison — *Relabelling symmetry => Kelvin's theorem & conservation of potential vorticity*
- 2002 Holm, Marsden, Ratiu — *Euler-Poincaré theory HP asymptotics*

- Reversible mechanical systems obey Hamilton's least action principle
- Conservation laws result from a symmetry of the action
- Hamilton's principle asymptotics : *approximating the action instead of the equations of motion systematically produces approximate systems with all conservation laws*

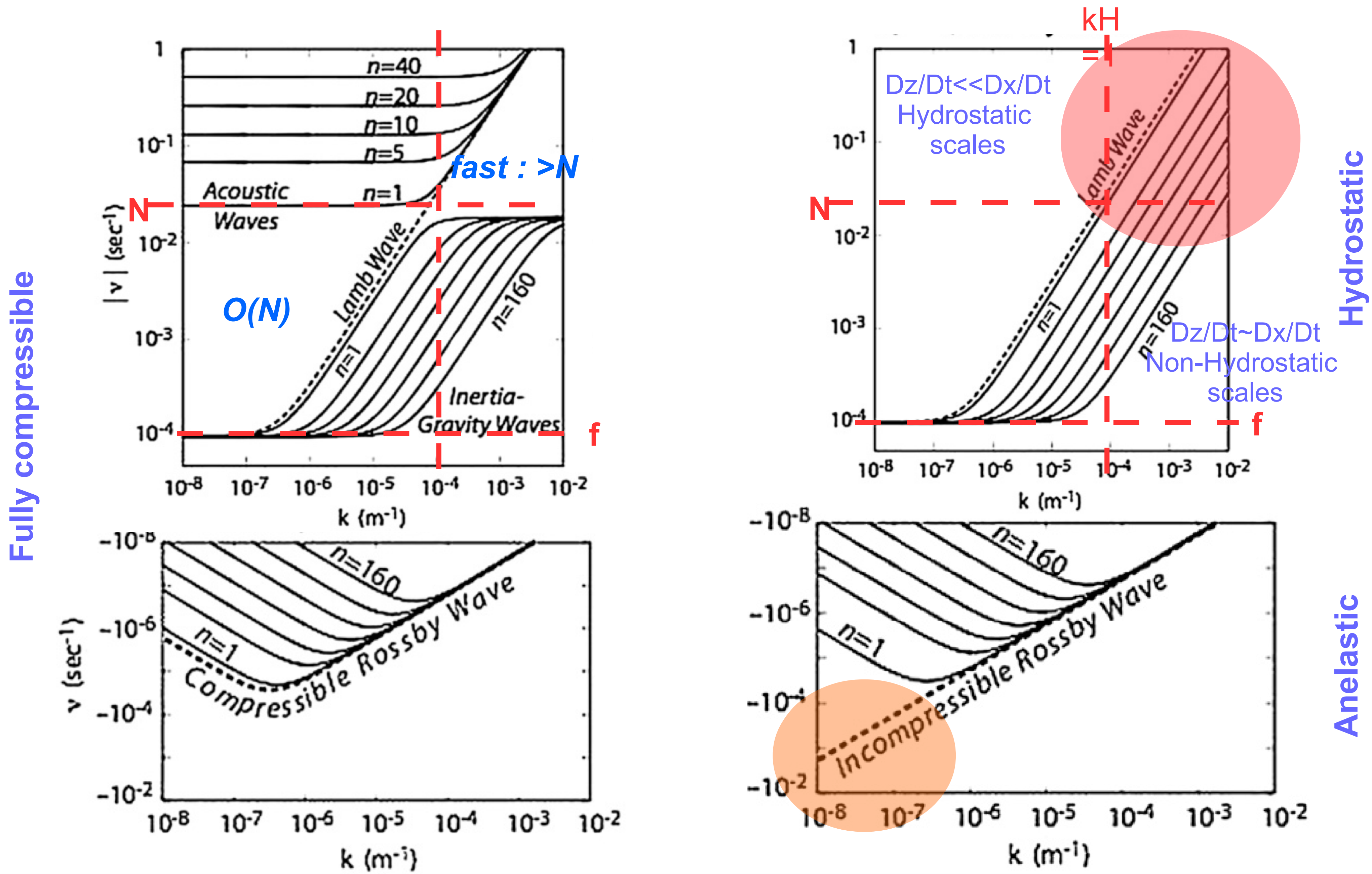
Least action principle in curvilinear coordinates

(Tort & Dubos, accepted by J. Atmos. Sci.)



Filtering acoustic waves : hydrostatic vs anelastic or Charybdis vs Scylla

(waves over isothermal atmosphere : e.g. Davies et al., 2003 ; Arakawa & Konor, 2009)



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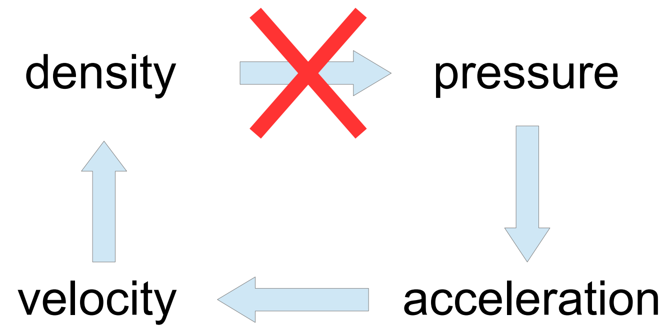
Getting rid of the acoustic waves : the « unified » way

Constraining (slaving) the density field

- breaks the pressure-density feedback loop
- key to suppressing acoustic waves

- « anelastic » slaving incorrect at large scales
- « hydrostatic » slaving is correct



Neglecting vertical acceleration implies hydrostatic balance
but the converse is not true.

Arakawa & Konor (2009)

- impose density through hydrostatic balance but retain vertical acceleration
- pressure is the sum of hydrostatic pressure and a non-hydrostatic deviation
- non-hydrostatic pressure determined from a Poisson-like problem

Variational implementation of this physical idea ?

- hydrostatic balance is a holonomic constraint (velocity not involved)
- Let us introduce a Lagrange multiplier to impose it !

$$
\begin{aligned}
\mathcal{L} \quad &= \quad \int \left[\frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{2} - gz - e\left(\frac{1}{\rho}, s\right) \right] \rho \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
& \quad - \int \left(\lambda \rho g + p_{qs} \frac{\partial \lambda}{\partial z} \right) \mathrm{d}x\mathrm{d}y\mathrm{d}z
\end{aligned}
$$

$$p_{qs} = p(\rho, s)$$

Dubos & Voitus (submitted)

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$$p_{qs} = p(\rho, s)$$

$$Z = z + \lambda$$

$$\mathcal{L} \simeq \int \left[\frac{\dot{x}^2 + \dot{y}^2 + \dot{Z}^2}{2} - gZ) - e\left(\frac{1}{\rho} \frac{\partial Z}{\partial z}, s\right) \right] \rho \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

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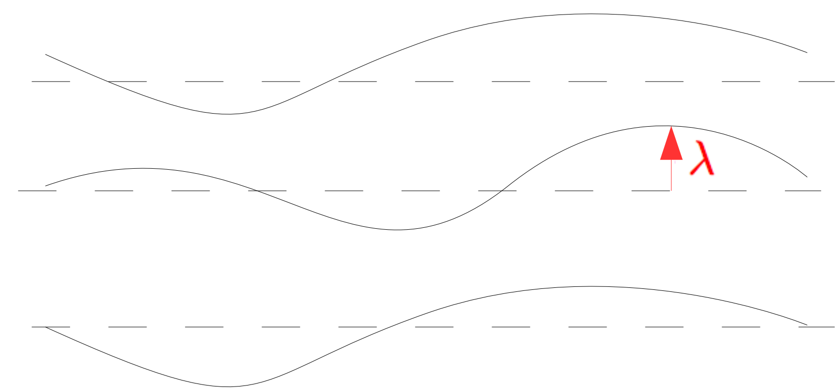
Interpretation

- The true height of air parcels is $Z = z + \lambda$
- z is their *hydrostatic height*
- λ is a vertical *non-hydrostatic displacement*

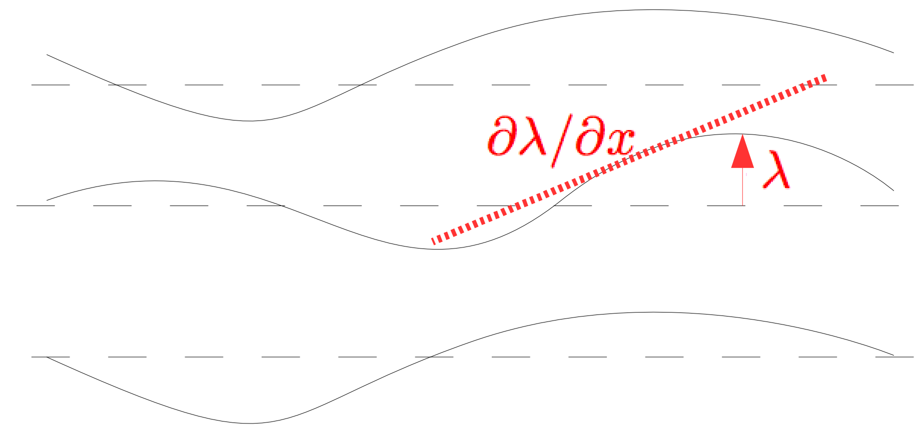


$$\mathcal{L} \simeq \int \left[\frac{\dot{x}^2 + \dot{y}^2 + \dot{Z}^2}{2} - gZ) - e\left(\frac{1}{\rho}\frac{\partial Z}{\partial z}, s\right) \right] \rho \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

Approximation is accurate if d*λ/dz << 1 and either*

- DZ/Dt<<Dx/Dt : hydrostatic scales
- or D*λ/Dt<<Dz/Dt :* hydrostatic velocity is an accurate estimate of true vertical velocity

Dubos & Voitus (submitted)



- The true height of air parcels is $Z = z + \lambda$
- z is their *hydrostatic height*
- *λ* is a vertical *non-hydrostatic displacement*

- coordinates (x,y,z) are slighty curvilinear
- ρ is the pseudo-density associated to (x,y,z)
- the true density and pressure are ρ+ρ' and p+p' :

$$\rho' = -\rho\frac{\partial\lambda}{\partial z}, \qquad p' = \rho'\frac{\partial p}{\partial \rho} = -\rho c^2\frac{\partial\lambda}{\partial z}$$

Advective form

Usual terms | *Non-hydrostatic pressure* | *Pseudo-forces due to (x,y,z) being curvilinear*

$$\underbrace{\frac{D\mathbf{u}}{Dt} + \theta\nabla\pi_{qs} + g\mathbf{e}_z}_{\text{Usual terms}} = -\theta\nabla\pi' \quad - g\nabla\lambda$$

Included in AK09 (for $-\theta\nabla\pi'$) — ***neglected by AK09*** (for $-g\nabla\lambda$)

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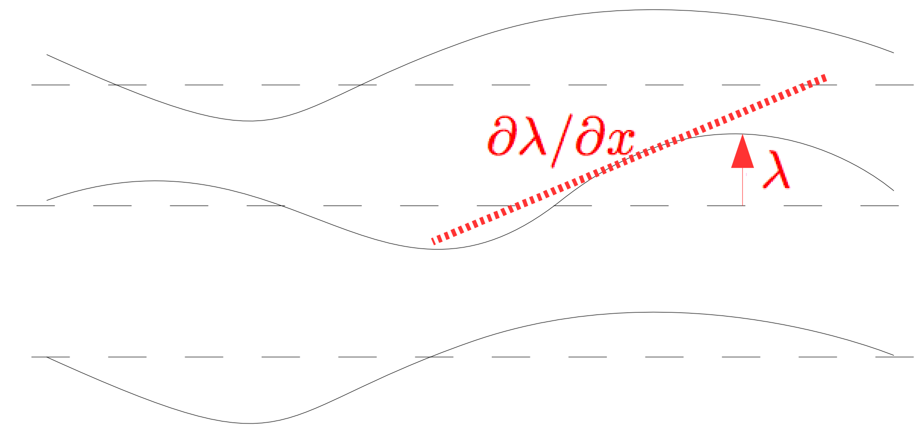

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Momentum budget

Usual terms — *Non-hydrostatic pressure* — *Pseudo-forces due to (x,y,z) being curvilinear*

$$\underbrace{\partial_t\left(\rho\mathbf{u}\right) + \mathrm{div}\left(\rho\mathbf{u}\otimes\mathbf{u}\right) + \nabla p_{qs} + \rho g\mathbf{e}_z}_{\text{Usual terms}} = \nabla\left(\rho c^2\frac{\partial\lambda}{\partial z}\right) - \nabla\left(p_{qs}\frac{\partial\lambda}{\partial z}\right) + \frac{\partial}{\partial z}\left(p_{qs}\nabla\lambda\right)$$

« Elliptic » problem for the non-hydrostatic displacement

- Assuming rigid boundaries, Dirichlet boundary conditions λ=0
- Assuming flat boundaries, Dirichlet boundary conditions w=0
- w=Dz/Dt obeys the same Richardson's equation as with hydrostatic equations
  (Richardson, 1922 ; see also Dubos & Tort, submitted to MWR)

$$\mathcal{A}\cdot w + \mathcal{B}\cdot \mathbf{u}_H = 0,$$

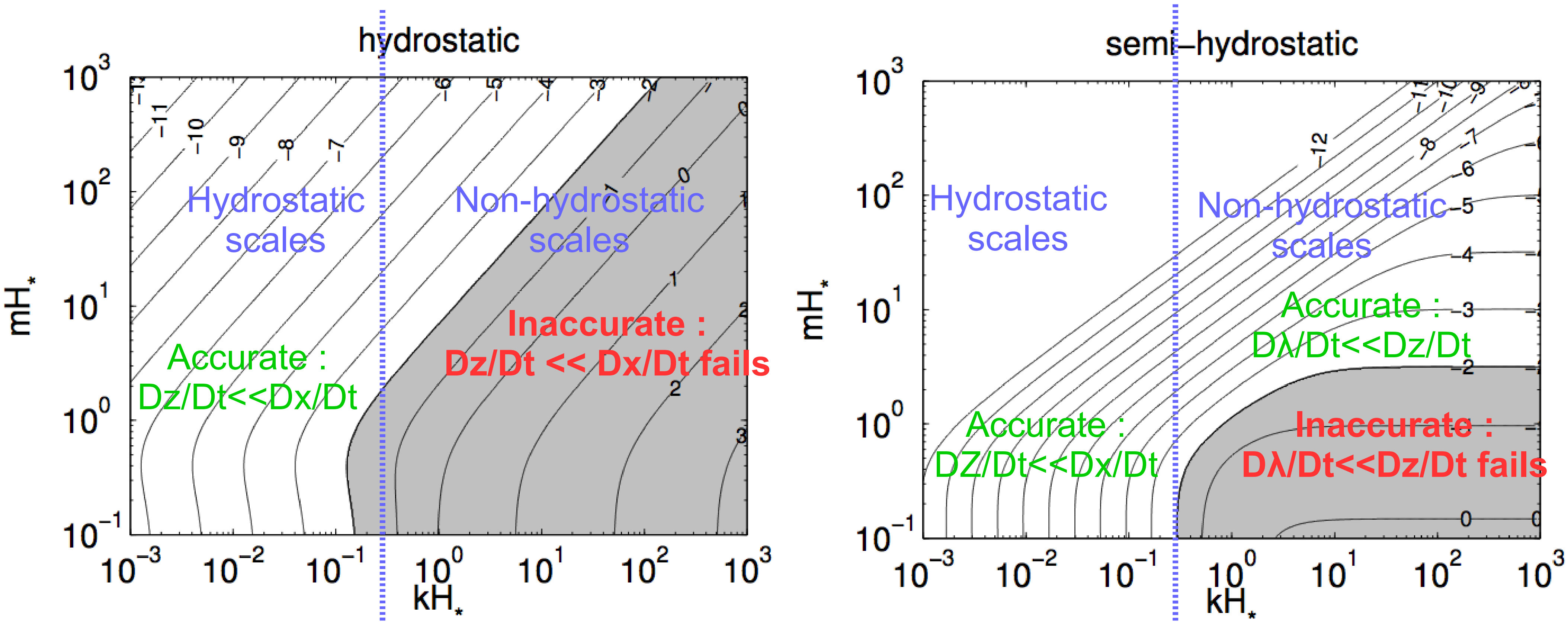
$$\begin{aligned}
\mathcal{A} \,:\, w &\mapsto \partial_z\left(\rho c^2 \partial_z w\right) \\
\mathcal{B} \,:\, \mathbf{u}_H &\mapsto \partial_z\left(\rho c^2 \partial_{\mathbf{x}}\cdot \mathbf{u}_H + \mathbf{u}_H\cdot\partial_{\mathbf{x}} p_{qs}\right) + g\partial_{\mathbf{x}}\cdot\left(\rho \mathbf{u}_H\right) \\
\mathcal{B}^* \,:\, \lambda &\mapsto \partial_{\mathbf{x}}\left(\rho c^2 \partial_z \lambda\right) - \partial_z\lambda\,\partial_{\mathbf{x}} p - \rho g \partial_{\mathbf{x}}\lambda
\end{aligned}$$

- Now observe that the momentum budget is of the form $\partial_t w - \frac{1}{\rho}\mathcal{A}\cdot\lambda = r.h.s.$
- One time-differentiation yields : $\partial_t \mathbf{u}_H - \frac{1}{\rho}\mathcal{B}^*\cdot\lambda = r.h.s.$

$$\begin{pmatrix} \mathcal{B}\frac{1}{\rho}\mathcal{B}^* & \mathcal{A} \\ \mathcal{A} & -\rho \end{pmatrix}\begin{pmatrix} \lambda \\ \partial_t w \end{pmatrix} = r.h.s$$

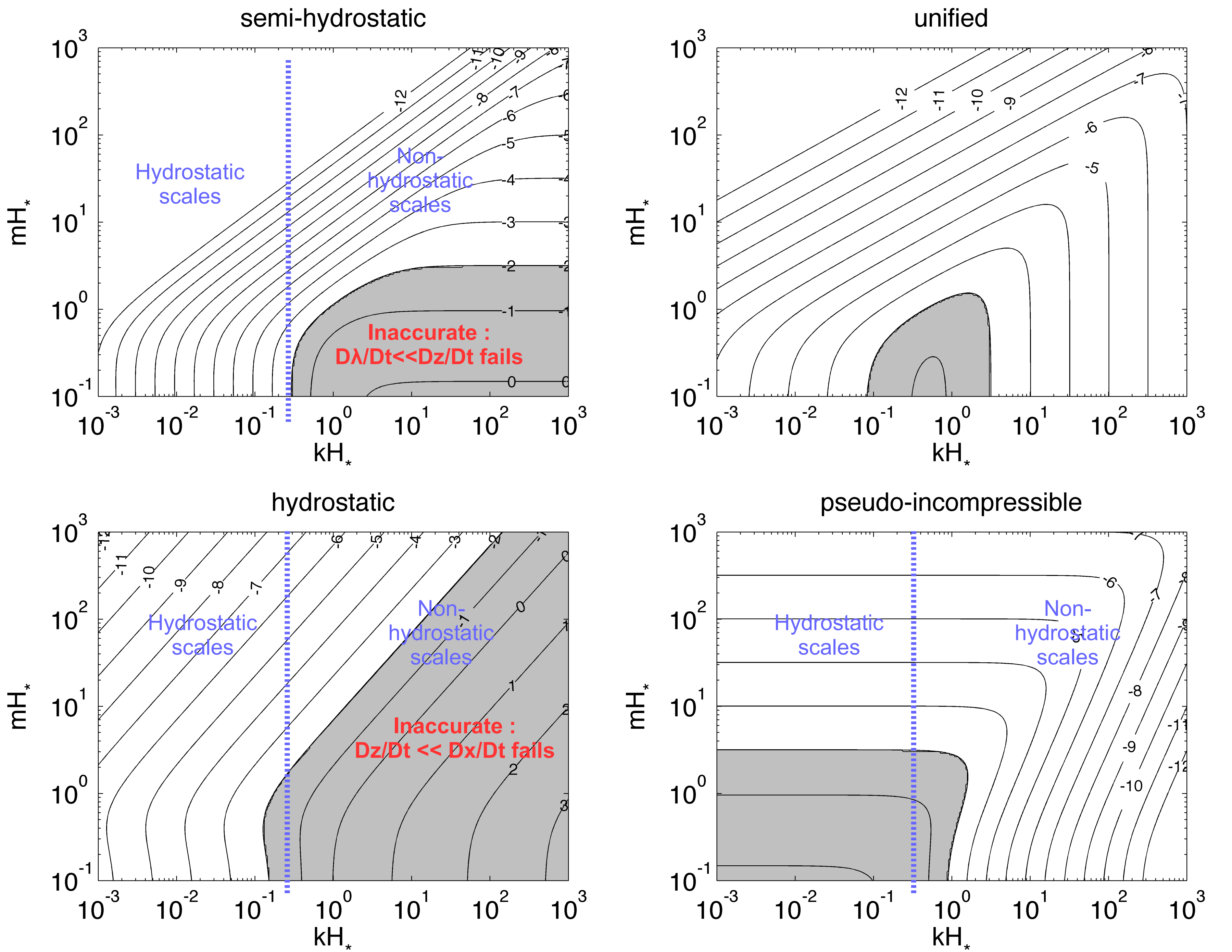
- *Not Poisson-like : hydrostatic constraint involves vertical derivative of density*
- *Information propagates horizontally no farther than ~ scale height*

Relative frequency error for waves in an isothermal atmosphere at rest



The semi-hydrostatic approximation corrects much of the errors present in the hydrostatic approximation, except for vertically-long waves (several scale heights)

Relative frequency error for waves in an isothermal atmosphere at rest



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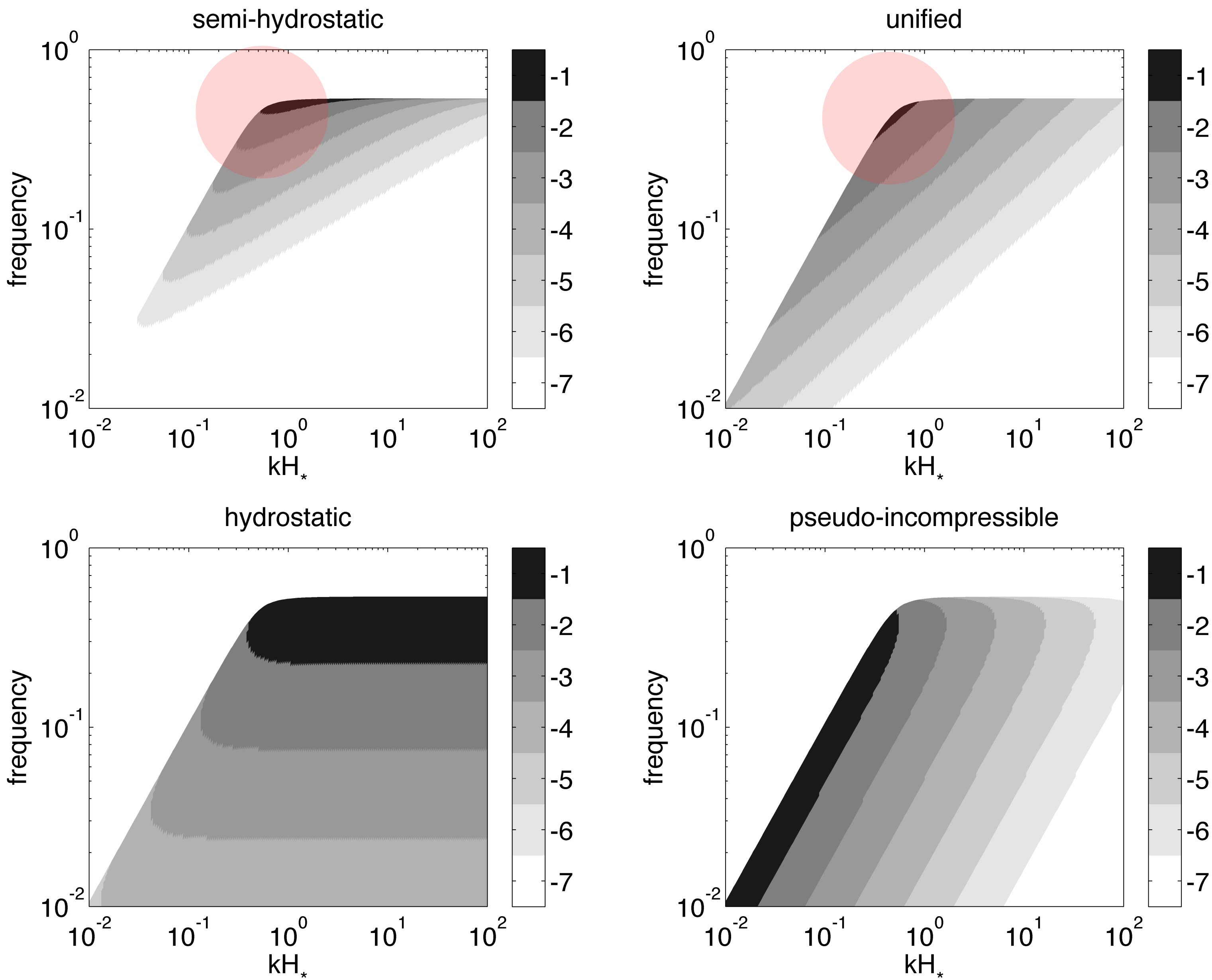

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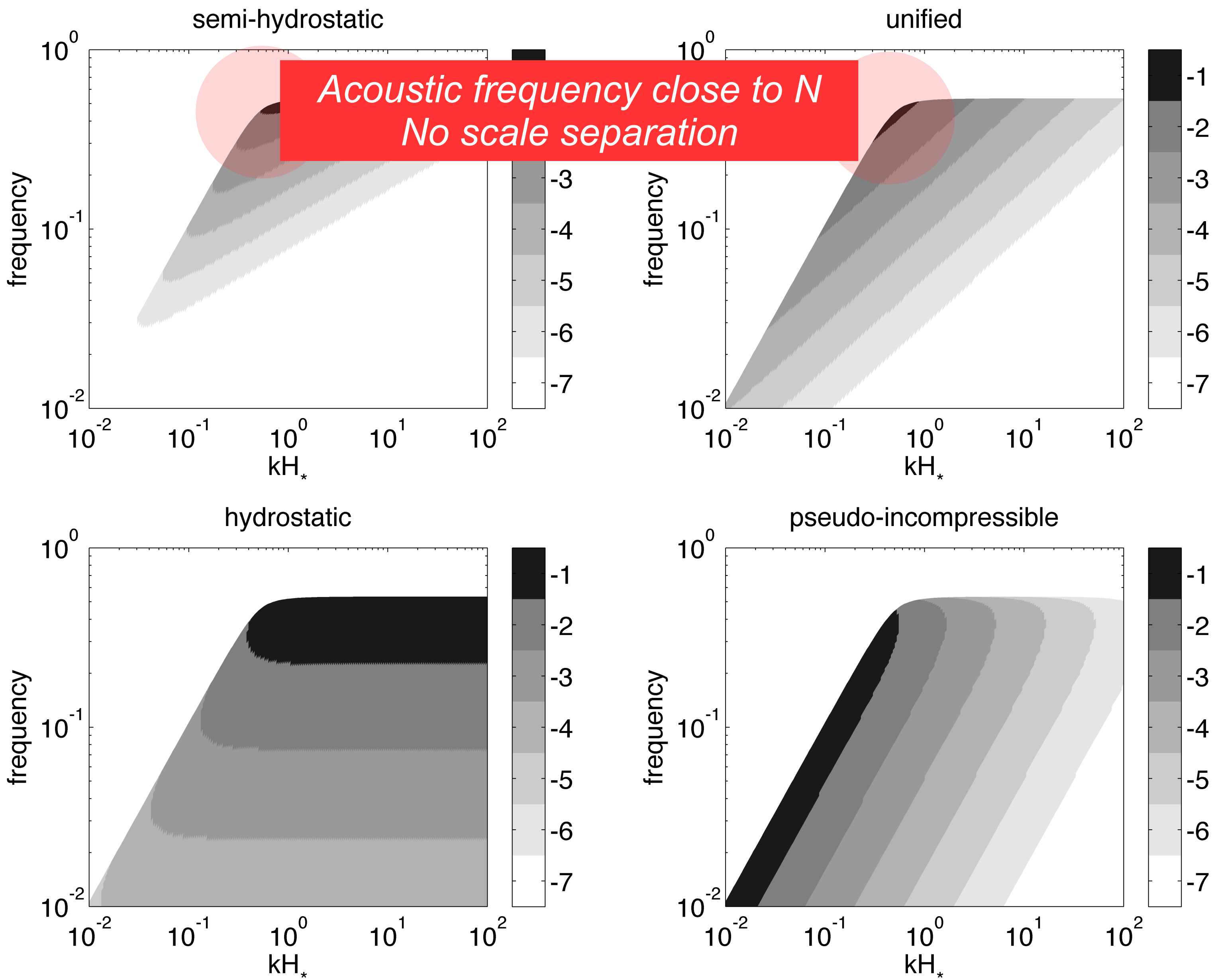

Laws of (inviscid) atmospheric motion can be expressed concisely
and safely from Hamilton's principle of least action

Variational principles can be helpful for

- Classifying existing approximation :

  Tort & Dubos (J. Atmos.Sci., accepted)
- Deriving new, consistent approximations

  Tort & Dubos (QJRMS, 2014), Dubos & Voitus (J. Atmos. Sci., submitted)

The semi-hydrostatic system :

- Conserves energy, momentum, potential vorticity
- Possesses a well-defined self-adjoint problem yielding NH pressure
- Is accurate from hydrostatic to NH scales
- Except horizontally short, vertically long gravity waves

- Has been derived for an arbitrary equation of state
- Can easily be extended to include : moisture, deep-atmosphere, etc .

Laws of (inviscid) atmospheric motion can be expressed concisely
and safely from Hamilton's principle of least action

Approximate laws of motion provide insight into

- Origin / nature of forces
- Actual, independant degrees of freedom
- 'Slaving' relationships between dependent and indepent DOFs

For atmospherically-relevant flow regimes, including non-hydrostatic,

the independent degrees of freedom of atmospheric motion
are precisely those of the hydrostatic primitive equations