State Key Laboratory of Numerical Modelling for Atmospheric Sciences
and Geophysical Fluid Dynamics(LASG)
Institute of Atmospheric Physics Chinese Academy of Sciences

PDEs on the Sphere, 2014

Boulder, CO, USA

Solutions of 3-D Coordinate Surfaces of an Orthogonal Terrain-Following Coordinate and its Preliminary 2-D Advection Experiments

Jinxi Li[1], Yiyuan Li[1], Bin Wang[1,2]

1 State Key Laboratory of Numerical Modeling for Atmospheric Sciences and Geophysical Fluid Dynamics, Institute of Atmospheric Physics, Chinese Academy of Sciences, Beijing, China, 100029

2 Ministry of Education Key Laboratory for Earth System Modeling, and Center for Earth System Science, Tsinghua University, Beijing, China, 100084

2014.4.7

Y. Li, B. Wang, D. Wang, J. Li, and L. Dong: An orthogonal terrain-following coordinate and its preliminary tests using 2-D Idealized advection experiments, 2014, *under review for GMD*.

Outline

- Background
- Solutions of 3-D Coordinate Surfaces of an Orthogonal Terrain-Following Coordinate
- 2-D Advection Experiments
- Conclusions

A. Background

A- Background

A.1 Importance of the numerical model

Atmospheric model

- Weather forecast
- Climate simulation
- …

Dynamical core

- One of the most important parts of an atmospheric model
- Prediction of wind, pressure and so on
- …

Vertical coordinate

- An essential part of dynamical core
- Choice of vertical coordinate is the most important aspect of designing a model
- …

Griffies et al. (2000); Steppeler et al. (2003); Ji et al. (2005); Andrew Staniforth (2006); Staniforth and Wood (2008)

A- Background

A.2 The development of vertical coordinate

Z coordinate – Richardson (1922) intuitive in the real space

P coordinate - Sutcliffe and Godart (1947) continuous equation → diagnostic equation

Θ coordinate - Shapiro et al. (1973) eliminate the computational errors of vertical motion

Coordinate plane intersects with the terrain → initialize the grid above the terrain → computational errors

Terrain-following σ coordinate Low boundary → coordinate surface

- Phillips (1957); Gal-Chen and Somerville (1975)

Computational errors of pressure gradient and advection

A- Background

A.3 The problems of sigma coordinate

Pressure gradient error

- Interpolation method
- 2003
- Transformation method
  - Li et al., 2012

Advection error

- Hybrid σ-coordinate
- Orthogonal sigma coordinate (OS-coordinate)
  - Li et al., 2013

Non-orthogonal vertical grid

- Non-orthogonal transformation
- following coordinate in 2-D
  - Sharman et al.,1988

Our research:

✧3-D solutions of coordinate surfaces of OS-coordinate

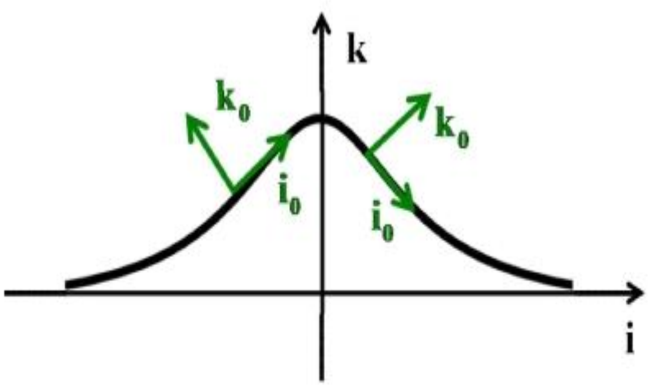
✧Compare the OS-coordinate with the hybrid σ-coordinate

B.

Solutions of 3-D Coordinate Surfaces of an Orthogonal Terrain-Following Coordinate

B- Solutions of 3-D Coordinate Surfaces of OS-coordinate

B.0 Design of an orthogonal terrain-following coordinate



2-D schematic of the coordinate rotation

$$\boxed{\begin{aligned} i_0 &= \cos b\lambda \cdot i - \sin b\theta \sin b\lambda \cdot j - \cos b\theta \sin b\lambda \cdot k \\ j_0 &= \cos b\theta \cdot j - \sin b\theta \cdot k \\ k_0 &= \sin b\lambda \cdot i + \sin b\theta \cos b\lambda \cdot j + \cos b\theta \cos b\lambda \cdot k \end{aligned}}$$

Basis vectors in OS-coordinate

Equations of each three coordinate surface:

$$\left\{\begin{aligned} \frac{\partial x'}{\partial x} &= \cos b\lambda \\ \frac{\partial x'}{\partial y} &= -\sin b\theta \sin b\lambda \\ \frac{\partial x'}{\partial z} &= -\cos b\theta \sin b\lambda \end{aligned}\right. \qquad \left\{\begin{aligned} \frac{\partial y'}{\partial x} &= 0 \\ \frac{\partial y'}{\partial y} &= \cos b\theta \\ \frac{\partial y'}{\partial z} &= -\sin b\theta \end{aligned}\right. \qquad \left\{\begin{aligned} \frac{\partial \sigma}{\partial x} &= \sin b\lambda \\ \frac{\partial \sigma}{\partial y} &= \sin b\theta \cos b\lambda \\ \frac{\partial \sigma}{\partial z} &= \cos b\theta \cos b\lambda \end{aligned}\right.$$

$$\cos\theta = \frac{1}{\sqrt{1+H_y^2}}, \cos\lambda = \frac{\sqrt{1+H_y^2}}{\sqrt{1+H_x^2+H_y^2}}, \cos\theta' = \frac{1}{\sqrt{1+H_x^2}}, \cos\lambda' = \frac{\sqrt{1+H_x^2}}{\sqrt{1+H_x^2+H_y^2}}$$

Li et al. (2013)

B- Solutions of 3-D Coordinate Surfaces of OS-coordinate

B.1 Idea of solving 3-D coordinate surfaces

$$\begin{cases} \dfrac{\partial \sigma}{\partial x} = \sin b\lambda \\ \dfrac{\partial \sigma}{\partial y} = \sin b\theta \cos b\lambda \\ \dfrac{\partial \sigma}{\partial z} = \cos b\theta \cos b\lambda \end{cases}$$

1. Construct the partial differential equations (PDEs) of each coordinate surfaces

Forward difference

2. Discrete the previous PDEs to obtain the linear algebraic equations (LAEs)

Generalized Least squares

3. Obtain the numerical solutions of LAEs, which is exactly the coordinate surfaces

B- Solutions of 3-D Coordinate Surfaces of OS-coordinate

B.2 An example of solving the sigma coordinate surface

PDEs

$$\begin{cases} \dfrac{\partial\sigma}{\partial x} = \sin b\lambda \\ \dfrac{\partial\sigma}{\partial y} = \sin b\theta \cos b\lambda \\ \dfrac{\partial\sigma}{\partial z} = \cos b\theta \cos b\lambda \end{cases}$$

Forward difference

LAEs

$$\begin{cases} \dfrac{\sigma_{i+1,j,k} - \sigma_{i,j,k}}{\Delta x} = \sin(b\lambda)_{i,j,k} \\ \dfrac{\sigma_{i,j+1,k} - \sigma_{i,j,k}}{\Delta y} = \sin(b\theta)_{i,j,k} \cos(b\lambda)_{i,j,k} \\ \dfrac{\sigma_{i,j,k+1} - \sigma_{i,j,k}}{\Delta z} = \cos(b\theta)_{i,j,k} \cos(b\lambda)_{i,j,k} \end{cases}$$

Matrix Form

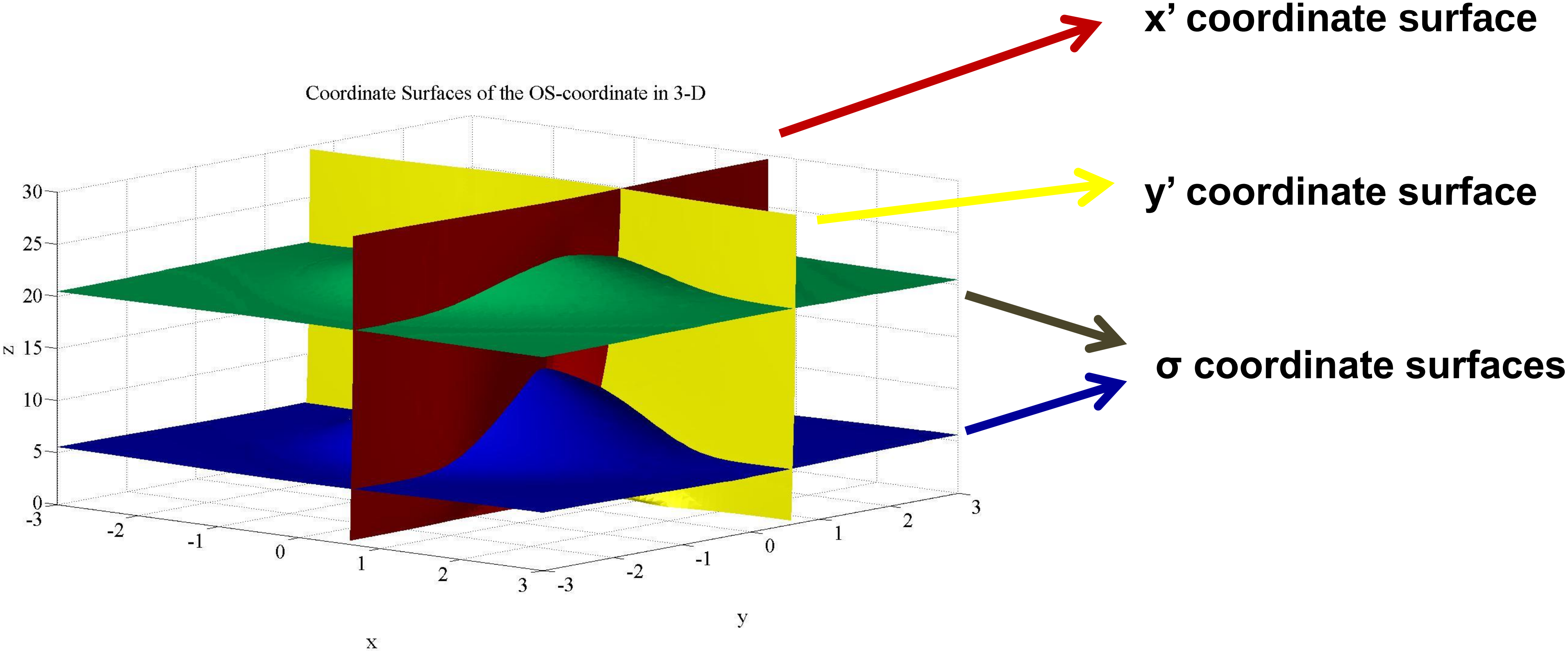
$$\boxed{A\sigma = B_3\left(\lambda_{i,j,k}, \theta_{i,j,k}\right)}$$

Generalized least squares

$$\sin\theta = \frac{H_y}{\sqrt{1+H_y^2}}, \sin\lambda = \frac{H_x}{\sqrt{1+H_x^2+H_y^2}}$$

B- Solutions of 3-D Coordinate Surfaces of OS-coordinate

B.3 Coordinate surfaces of the OS-coordinate in 3-D



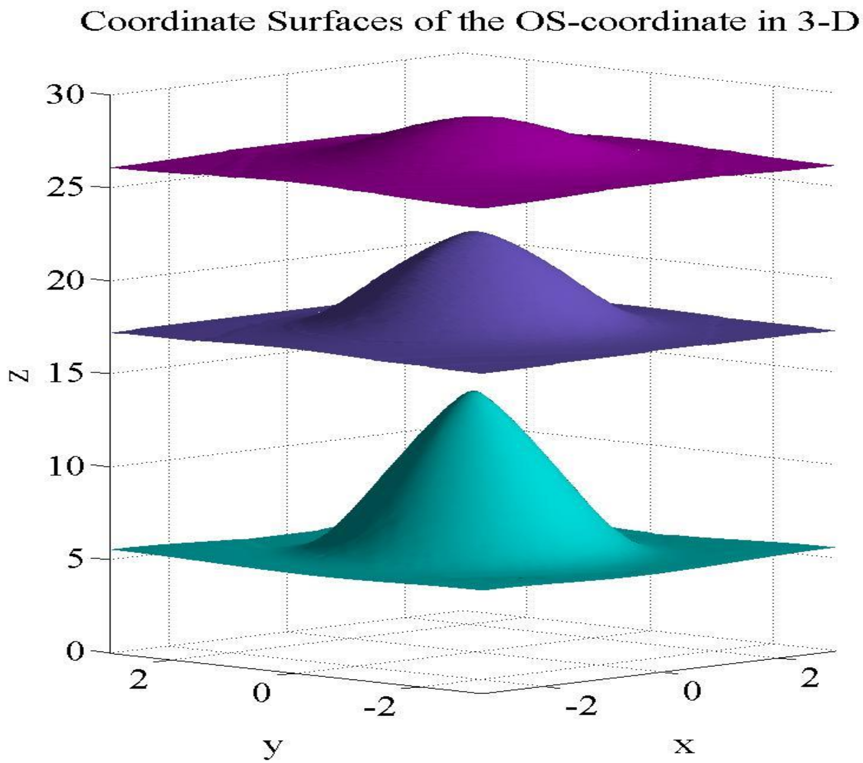
Conclusion:

1.Each two coordinate surfaces intersect orthogonally;

2.Intersection lines between every two coordinate surfaces are curves instead of lines.

B- Solutions of 3-D Coordinate Surfaces of OS-coordinate

B.4 Vertical variation of the σ-levels in the OS-coordinate



Conclusion:

Preserve the benefits of classic sigma coordinate

- **Bottom σ-level coincides with the terrain**
- **Top σ-level becomes flat at the top of the model**
- **Slopes of the σ-levels decrease with increasing height**

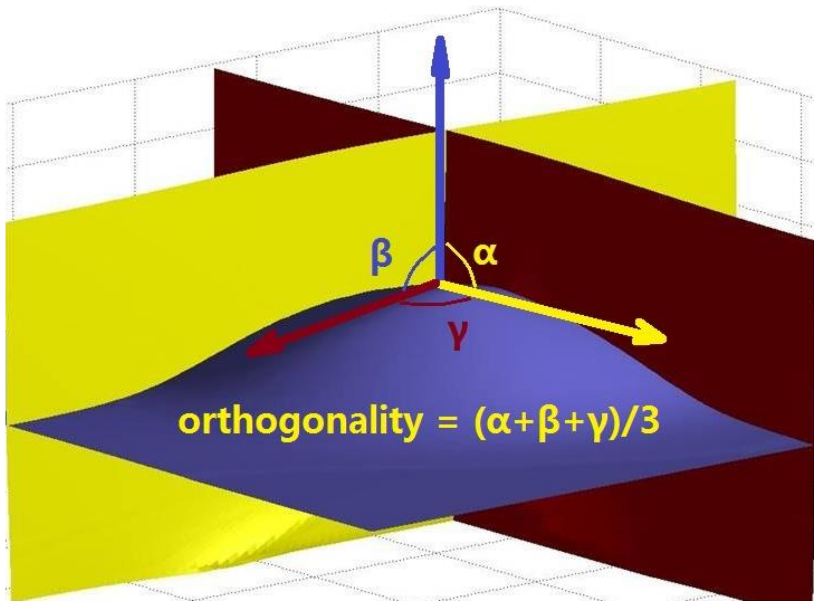
B- Solutions of 3-D Coordinate Surfaces of OS-coordinate

B.5 The orthogonality of 3-D numerical solutions of coordinate surfaces

Obtain *the normal vectors* of each coordinate surface via every grid point

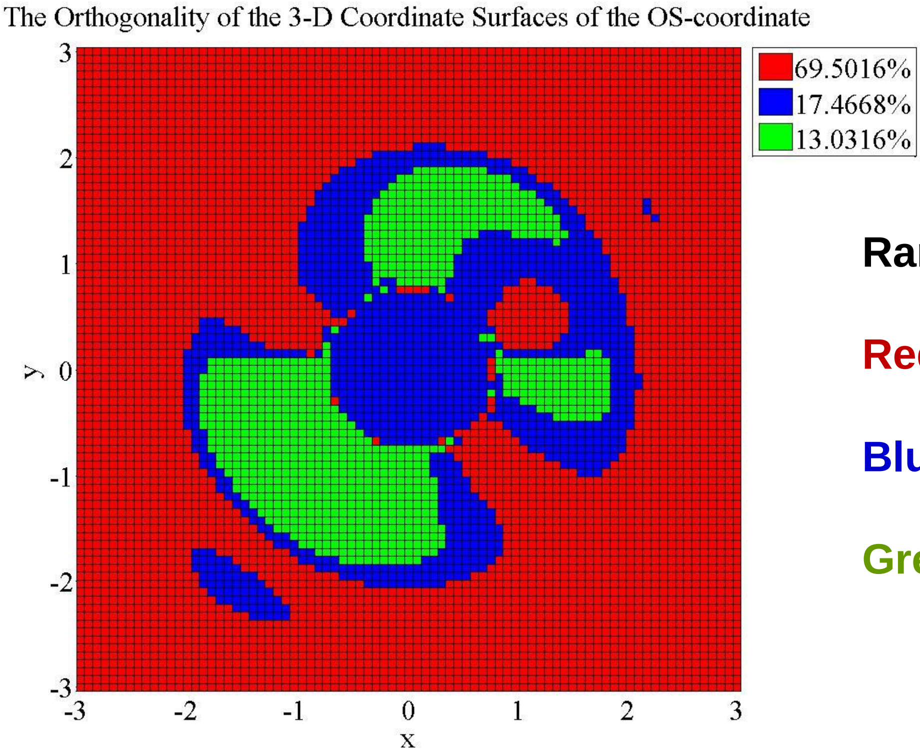
Calculate *the intersection angle* between each two normal vectors

Calculate *the orthogonality* using those intersection angles



B- Solutions of 3-D Coordinate Surfaces of OS-coordinate

B.6 The orthogonality of the points on the coordinate surfaces of the OS-coordinate at the same height



Range of angle in different colors：

Red: 80-100º

Blue: 70-80 º or 100-110 º

Green: 60-70 º or 110-120 º

Conclusion:

1. 70% of all the points on a coordinate surfaces is quasi-orthogonal.

2.The most non-orthogonal angles appear above the steepest terrain.

B- Solutions of 3-D Coordinate Surfaces of OS-coordinate

B.7 A summary in short

We can do it	• Solve out the solutions of every coordinate surfaces in OS-coordinate in 3-D
But it is not good enough now	• Only 70% of points is nearly orthogonal at present • Most non-orthogonal angles appear above the steepest terrain
However it can be improved	• Modify the PDEs of each coordinate surface • Use other discretization method to obtain the LAEs • Use high-order methods to solve the LAEs

C.

2-D Advection Experiments

C- 2-D Advection Experiments

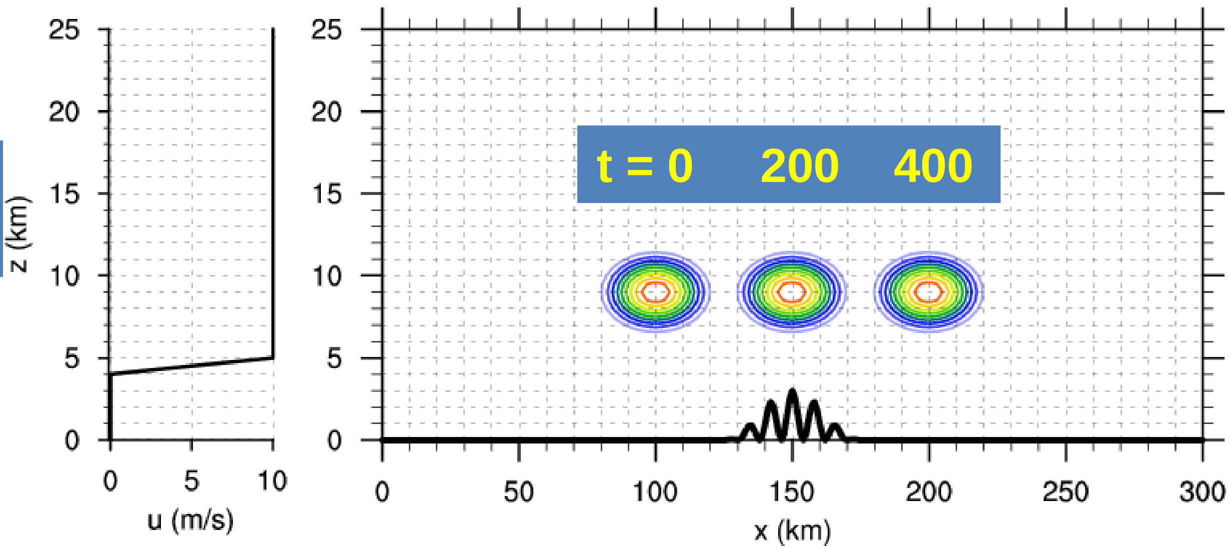
C.1 Schar-type 2-D linear advection experiments

(reproduce the experiments designed by Schär et al., 2002)

- **OS-coordinate VS the corresponding hybrid σ-coordinate**
- **Wavelike terrain**

$$\boxed{\frac{q_{i,k}^{n+1}-q_{i,k}^{n-1}}{2\Delta t}+u'\,\frac{q_{i+1,k}^{n}-q_{i-1,k}^{n}}{2\Delta X}+w'\,\frac{q_{i,k+1}^{n}-q_{i,k-1}^{n}}{2\Delta Z}=0}$$

Horizontal wind *u*



Analytical solution

The colored contours in the right panel represent the tracer *q* with the contour interval of 0.1, and the thick black curve, the wavelike terrain.

C- 2-D Advection Experiments

C.2 Computational mesh in OS-coordinate and corresponding hybrid σ-coordinate

Two sets of comparable experiments:

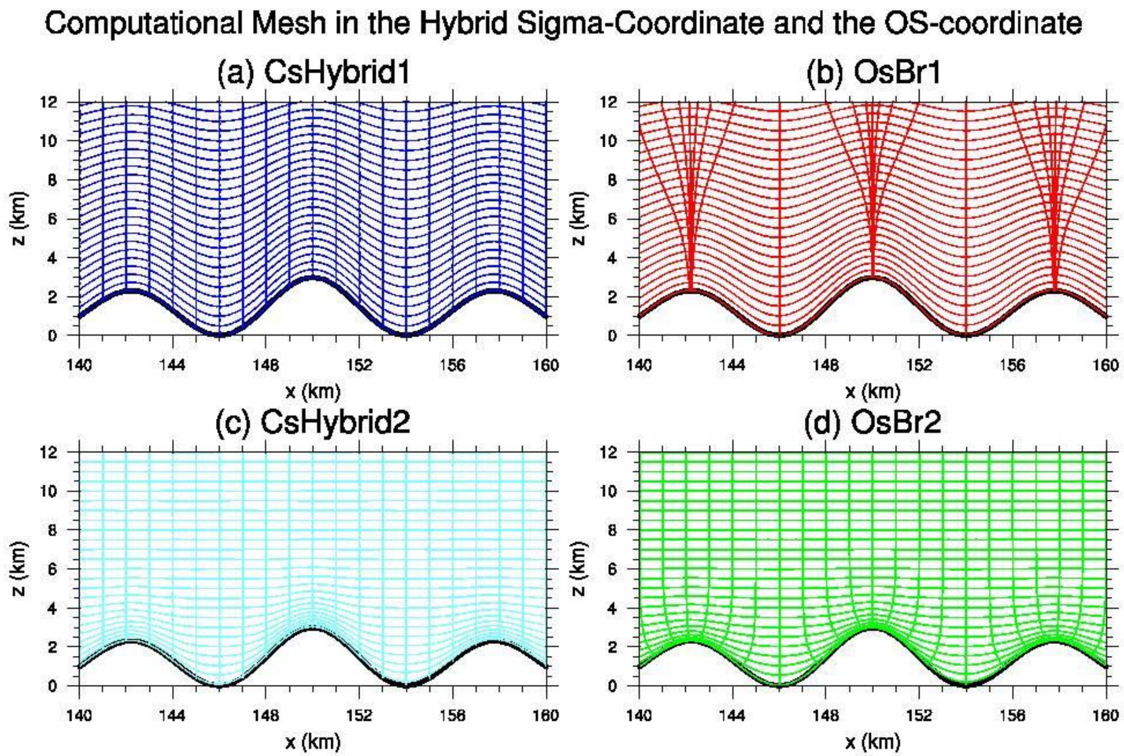
(*similar slopes* of vertical layers in the same row)

First row: steep slope of vertical layers

CsHybrid1, OsBr1

Second row: smooth slope of vertical layers

CsHybrid2, OsBr2



Comparison followed the three aspects:

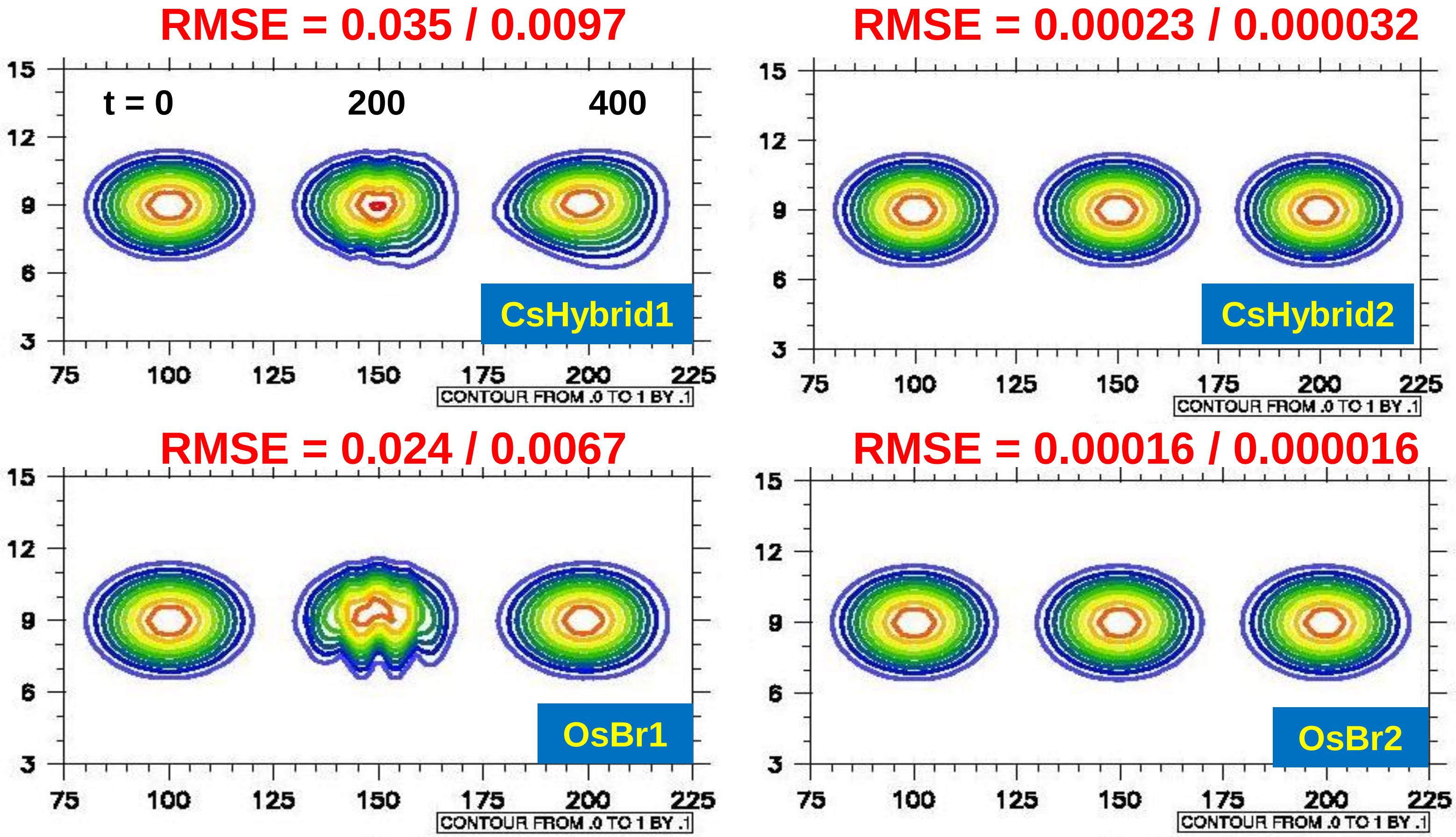
- **The numerical solutions**
- **Root mean square errors (RMSEs)**
- **RMSEs reduction by the OS-coordinate**

C- 2-D Advection Experiments

C.3 Numerical Solutions at three times

Hybrid σ-coordinate

OS-coordinate



Conclusion:

Colored contours are the tracer *q*, with the contour interval of 0.1.

1. RMSE in OS-coordinate is smaller than that in corresponding hybrid σ-coordinate
2. At the end of the advection, the shape of the tracer in CsHybrid1 still has a large deformation; the shape of the tracer in OsBr1 is almost recovered.

C- 2-D Advection Experiments

C.4 RMSEs of all five coordinates

Black :

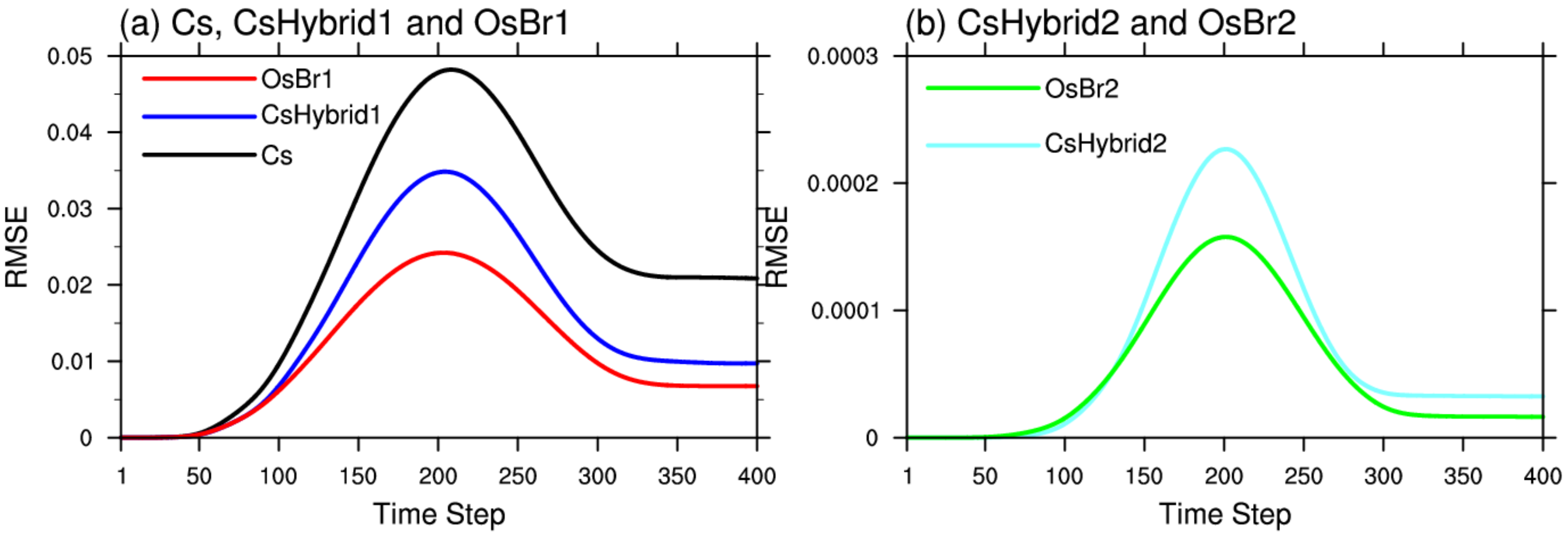
Classic σ-coordinate

Blue and sky-blue :

hybrid σ-coordinate

Red and green :

OS-coordinate



Conclusion:

The RMSEs in the OS-coordinate is much smaller compared with the corresponding hybrid σ-coordinate.

C- 2-D Advection Experiments

C.5 RMSEs reduction by the OS-coordinate

Experiments	RMSEs		RMSEs reduction by the OS-coordinate	
	average	maximum	average	maximum
CsHybrid1	0.015	0.035	26.9%	30.5%
OsBr1	0.011	0.024		
CsHybrid2	0.000068	0.00023	25.5%	30.4%
OsBr2	0.000051	0.00016		

C- 2-D Advection Experiments

C.6 A summary in short

	Conclusion (compared with the corresponding hybrid σ-coordinate)
Numerical solutions	The shape of the tracer in OS-coordinate can be preserved at the end of the advection
RMSE	Much smaller in OS-coordinate

D.

Conclusion

D- Conclusion

D.1 Solutions of 3-D coordinate surfaces of OS-coordinate

We can do it

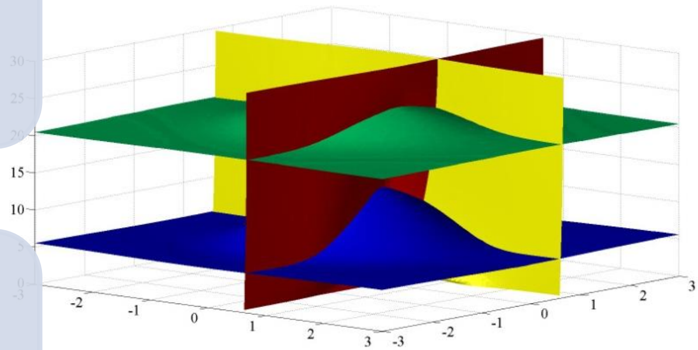
- Solve out the solutions of coordinate surfaces in OS-coordinate in 3-D

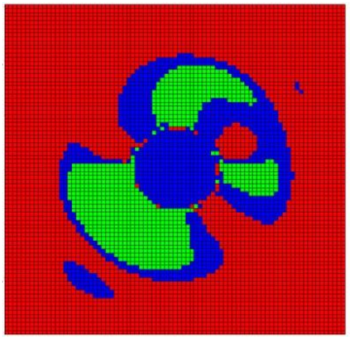
But it's not good enough now

- Only 70% of points is nearly orthogonal at present
- Most non-orthogonal angels are above the steepest terrain

However it can be improved

- Modify the PDEs
- Use other discretization method to obtain the LAEs
- Use high-order method to solve the LAEs





D- Conclusion

D.2 2-D advection experiments

1. The RMSEs in the OS-coordinate are much smaller than those of the corresponding hybrid σ-coordinate. The RMSEs reduction of the advection errors by the OS-coordinate is over 25% more.

2. The OS-coordinate can preserve the shape of the tracer much better than the hybrid σ-coordinate at the end of the advection.

Thank you!

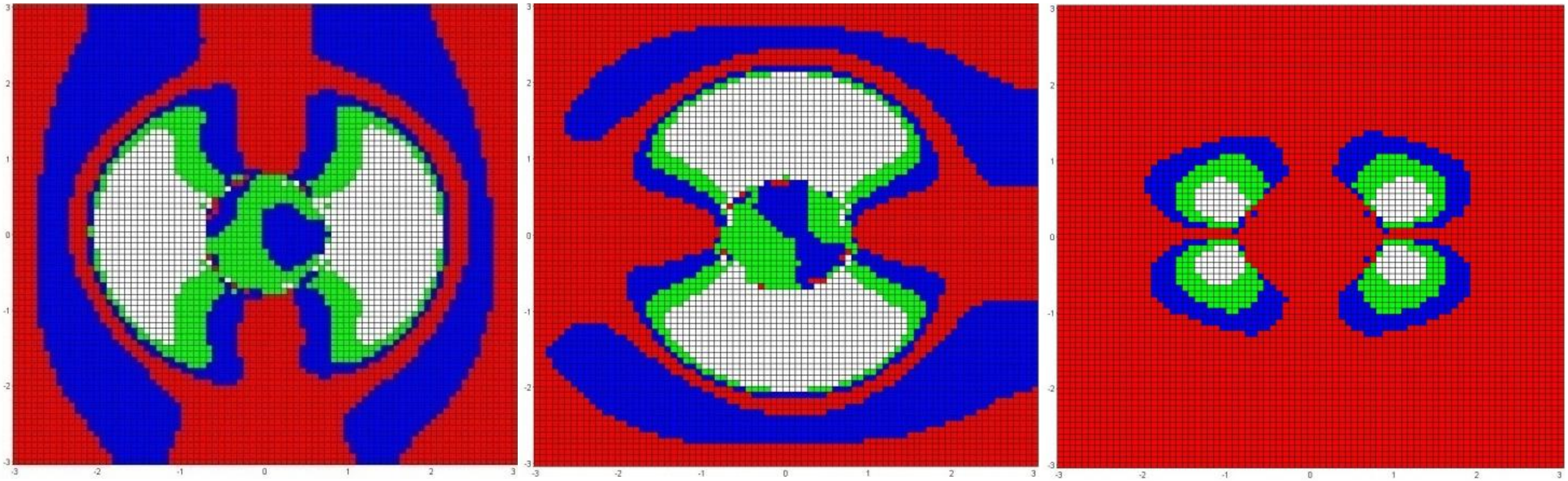
Special Thanks to the editor and three anonymous reviewer of GMD!

THANKS

LASG

THANKS

B.4 Result of angles between every two coordinate surface



α (x' & sigma) β (y' & sigma) γ (x' & y')

The orthogonality of the points on the coordinate surfaces of the OS-coordinate at a constant height.

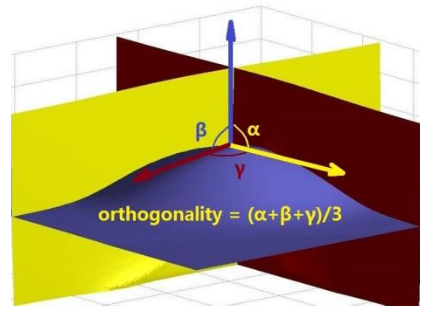
Range of angle in different colors：

Red: 80-100º

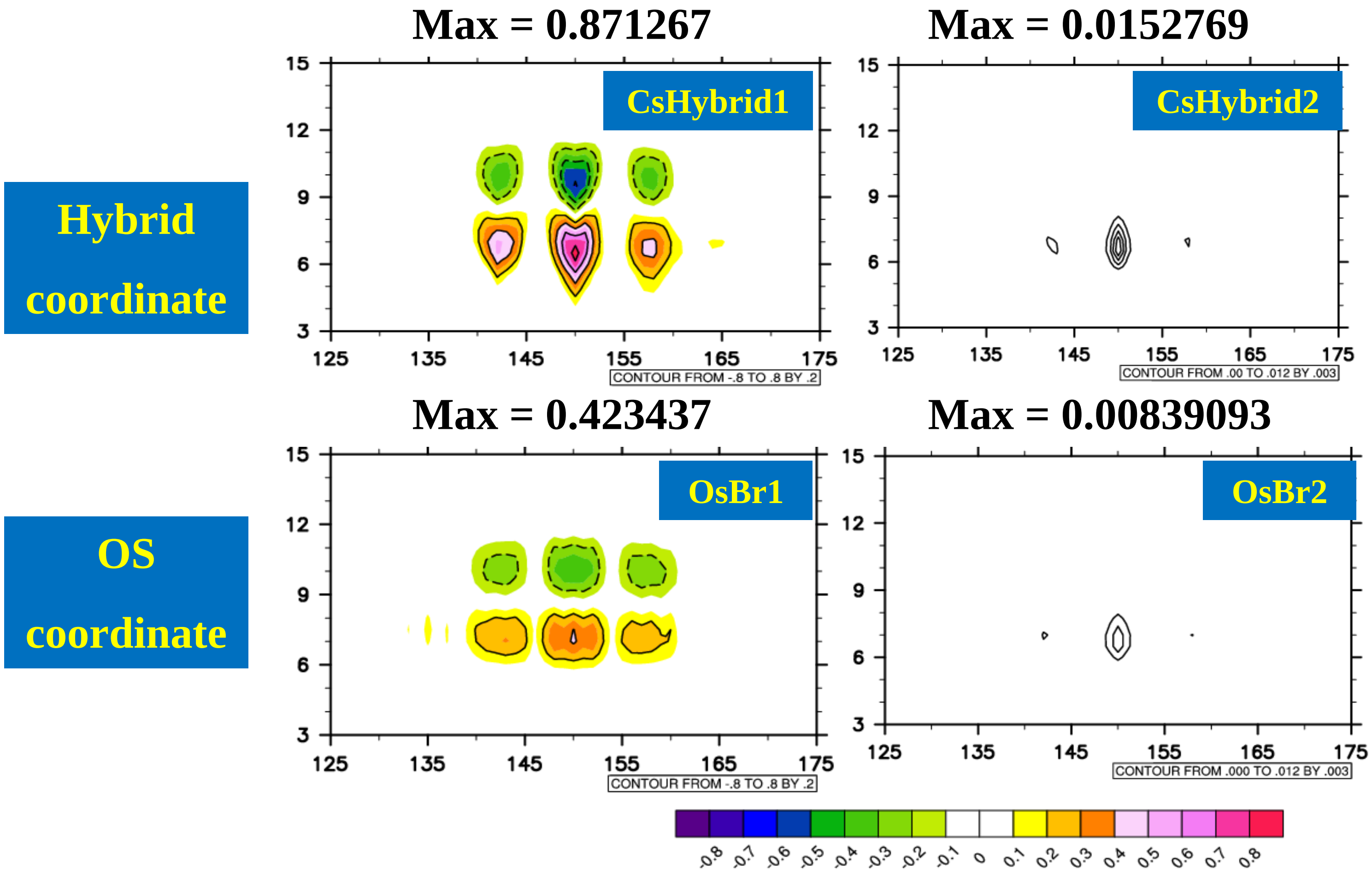
Blue: 70-80 º or 100-110 º

Green: 60-70 º or 110-120 º

White: other angels



C.4 Absolute errors in the Schär-type experiments



Absolute errors with the non-terrain simulation at *t* = 200

The solid black contours are for positive values, and the dashed contours are for negative values. The contour interval is 0.2.

Conclusion:

1.The absolute errors in OS-coordinate are much smaller than those in the hybrid σ-coordinate.

2.The maximum absolute error in OsBr2 is two orders of magnitude less than those in OsBr1

Further research

3D Solution

- Obtain PDEs: Seek the rotation angle of the vectors in the tangent plane
- Obtain LAEs: Change the difference methods
- Numerical Solutions: Change the numerical methods(generalized least square and conjugated gradient method)

Orthogonal Grid

- Investigation
- Analysis of mechanism

A- Background

A.2 The development of vertical coordinate

Z coordinate-Richardson(1922) — intuitive in the real space

P coordinate-Sutcliffe and Godart(1947) — continuous equation → diagnostic equation

Θ coordinate-Shapiro et al.(1973) — eliminate the computational errors of vertical advection

Coordinate plane intersects with the terrain → initialize the grid above the terrain → calculation errors

Terrain-following σ coordinate — Low boundary → coordinate surface

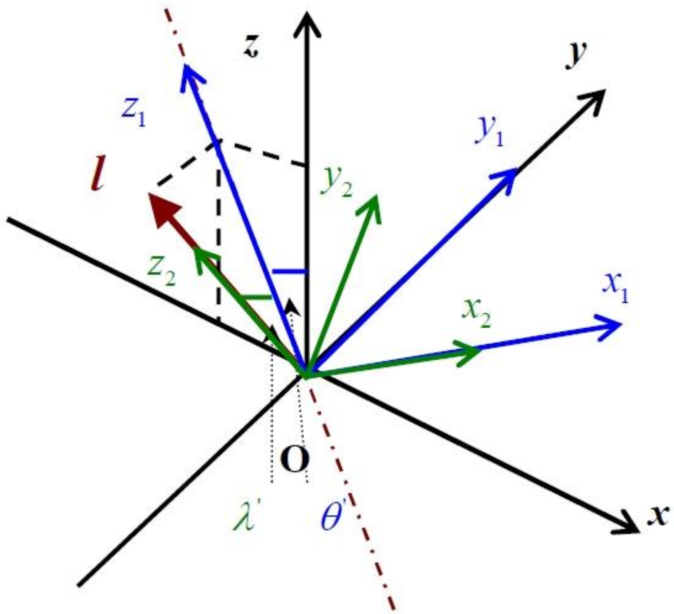
- Phillips(1957);Gal-Chen and Sommerville(1975)

Computational errors of pressure gradient force and advection

η coordinate — Quasi-horizontal vertical coordinate surface

- Mesinger(1984); Yu R. and Xu Y.,(2004)

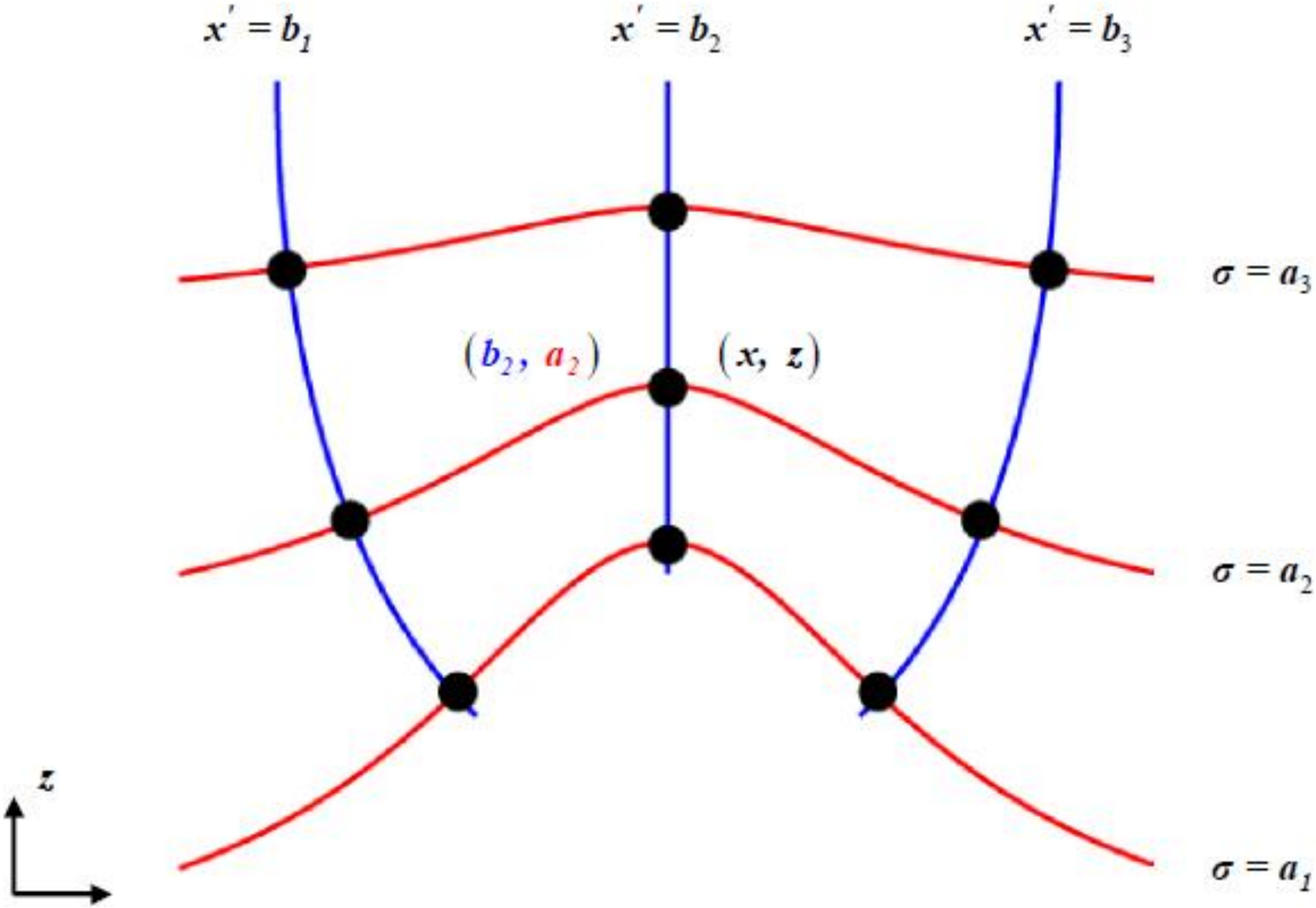
Rotation of OS-coordinate



3-D schematic rotation for solving the basis of the OS-coordinate on the upslope of the terrain.

The burgundy arrow is the normal vector of the terrain, and the burgundy dashdotted line is its projection on the plane Oxz. The black arrows are the basis vectors of the z coordinate, the blue arrows are the basis vectors of the first rotated coordinate [O;x1,y1,z1], and the green arrows are the basis vectors of the second rotated coordinate [O;x2,y2,z2].

Cross-Point Method



Calculate intersection points from the top of the model through x' coordinate line

Expressions of basis of OS-coordinate:

The first kind	$\begin{cases} \vec{\mathbf{i}}_o = \cos\theta' \vec{\mathbf{i}} + \sin\theta' \vec{\mathbf{k}} \\ \vec{\mathbf{j}}_o = -\sin\theta' \sin\lambda' \vec{\mathbf{i}} + \cos\lambda' \vec{\mathbf{j}} + \cos\theta' \sin\lambda' \vec{\mathbf{k}} \\ \vec{\mathbf{k}}_o = -\sin\theta' \cos\lambda' \vec{\mathbf{i}} - \sin\lambda' \vec{\mathbf{j}} + \cos\theta' \cos\lambda' \vec{\mathbf{k}} \end{cases}$
The second kind	$\begin{cases} \vec{\mathbf{i}}_o = \cos\lambda \vec{\mathbf{i}} - \sin\theta \sin\lambda \vec{\mathbf{j}} - \cos\theta \sin\lambda \vec{\mathbf{k}} \\ \vec{\mathbf{j}}_o = \cos\theta \vec{\mathbf{j}} - \sin\theta \vec{\mathbf{k}} \\ \vec{\mathbf{k}}_o = \sin\lambda \vec{\mathbf{i}} + \sin\theta \cos\lambda \vec{\mathbf{j}} + \cos\theta \cos\lambda \vec{\mathbf{k}} \end{cases}$

where $$\cos\theta = \frac{1}{\sqrt{1+H_y^2}}, \cos\theta' = \frac{1}{\sqrt{1+H_x^2}}, \cos\lambda = \frac{\sqrt{1+H_y^2}}{\sqrt{1+H_x^2+H_y^2}}, \cos\lambda' = \frac{\sqrt{1+H_x^2}}{\sqrt{1+H_x^2+H_y^2}}$$

Schär-type experiments (2002)

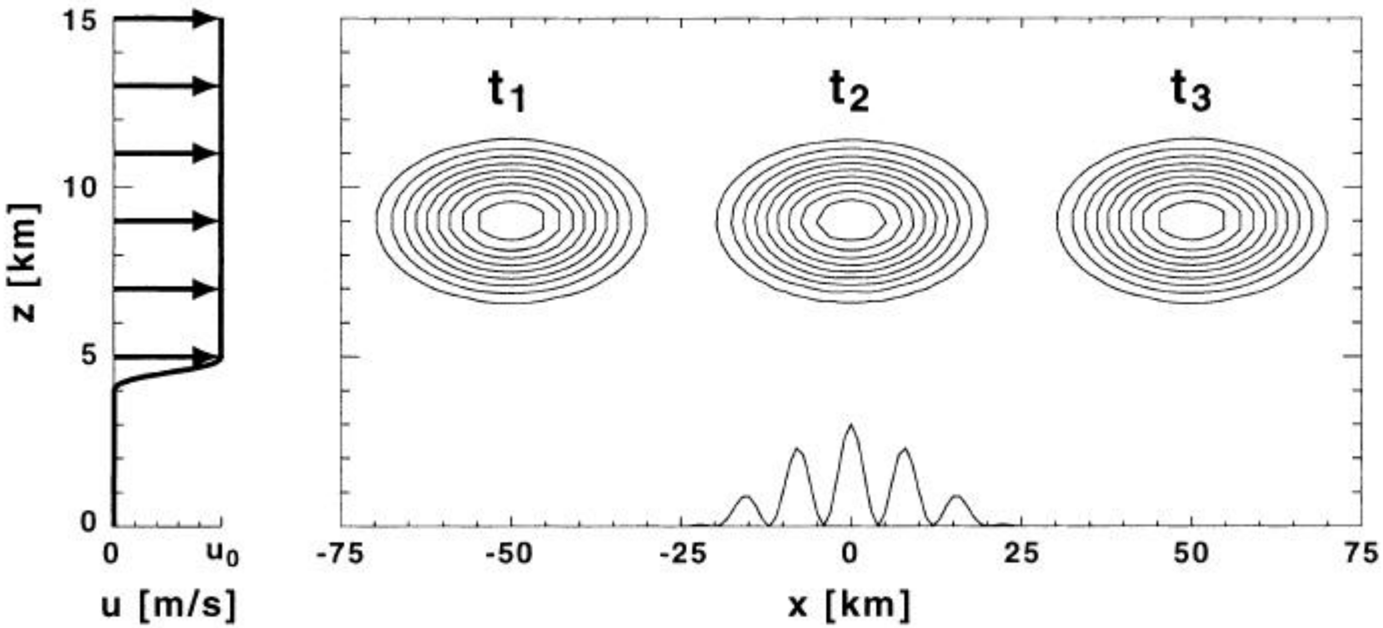
The new smooth level vertical (SLEVE) coordinate yields smooth coordinates at mid- and upper levels.

$$z(\sigma) = \sigma + h_1(x,y) b_1(\sigma) + h_2(x,y) b_2(\sigma)$$

$$b_i(\sigma) = \frac{sh \dfrac{H-\sigma}{s_i}}{sh \dfrac{H}{s_i}}$$

The basic concept of the new coordinate is to employ a scale-dependent vertical decay s of underlying terrain features.

Schär-type experiments (2002)



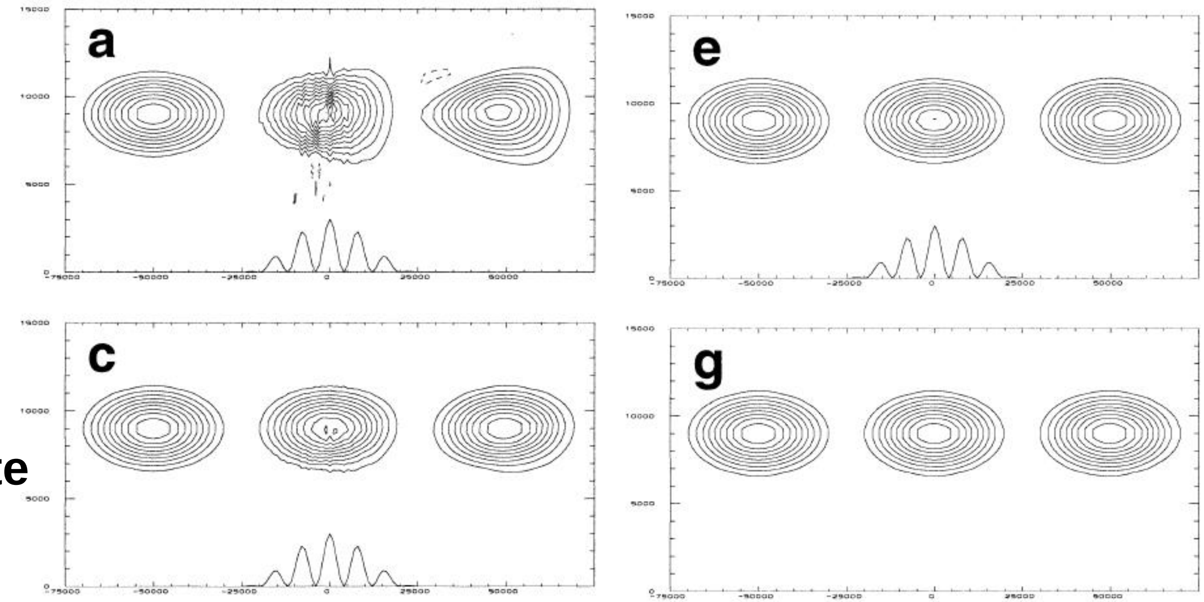
Vertical cross section of the idealized two-dimensional advection test.

The topography is located entirely within a stagnant pool of air, while there is a uniform horizontal velocity aloft. The analytical solution of the advected anomaly is shown at three instances.

Schär-type experiments (2002)

σ coordinate

SLEVE coordinate



Hybrid coordinate

A regular grid

Numerical solutions to the advection test using centered differences and a horizontal Courant number of 0.25.

(a),(c),(e),(g) The advected anomalies at three consecutive times (t1 = 0, t2 = 2500 s, t3 = 5000 s).

The initial amplitude of the anomaly is 1; the contour interval in the left-hand panels is 0.1, and that in the righthand panels is 0.01 (zero contour suppressed, negative contours dashed).

Parameters:

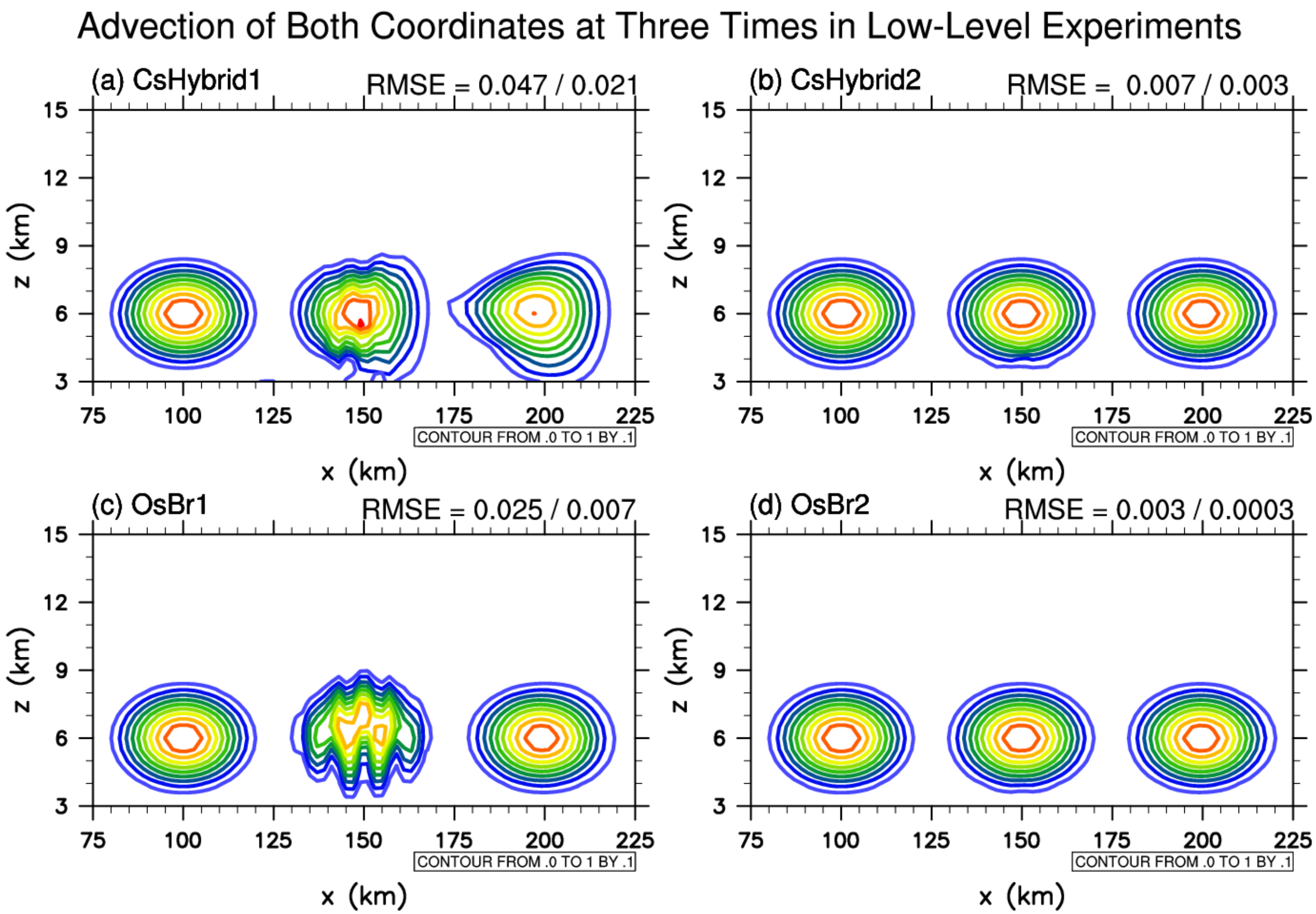
[OS-coordinate](#) VS [the associated hybrid σ coordinate](#)

Wave-like terrain

To investigate

the distinct impact of [the orthogonal grid](#) of the OS-coordinate

Experiment parameters	Expressions
wave-like terrain	$h^*(x)=\begin{cases} h_0\cos^2\left(\frac{\pi x}{2a}\right) & \text{for } \lvert x\rvert\le a \\ 0 & \text{for } \lvert x\rvert\ge a \end{cases}$
	$h(x)=\cos^2\left(\frac{\pi x}{\lambda}\right)h^*(x)$
Experiment parameters	$b=\left(\frac{H_t-z}{H_t-h}\right)^n$
definition of hybrid σ coordinate	$\sigma=z-\left(\frac{H_t-z}{H_t-h}\right)^n\cdot h$
advection equation	$\frac{q_{i,k}^{n+1}-q_{i,k}^{n-1}}{2\Delta t}+u'\frac{q_{i+1,k}^{n}-q_{i-1,k}^{n}}{2\Delta X}+w'\frac{q_{i,k+1}^{n}-q_{i,k-1}^{n}}{2\Delta Z}=0$
tracer	$q(x,z)=q_0\cdot\begin{cases}\cos^2\left(\frac{\pi}{2}\cdot r\right), & r\le 1\\ 0 & \end{cases}$
u field	$u(z)=u_0\cdot\begin{cases}1 & , z_2\le z\\ \sin^2\left(\frac{\pi}{2}\cdot\frac{z-z_1}{z_2-z_1}\right) & , z_1\le z\le z_2\\ 0 & , z\le z_1\end{cases}$



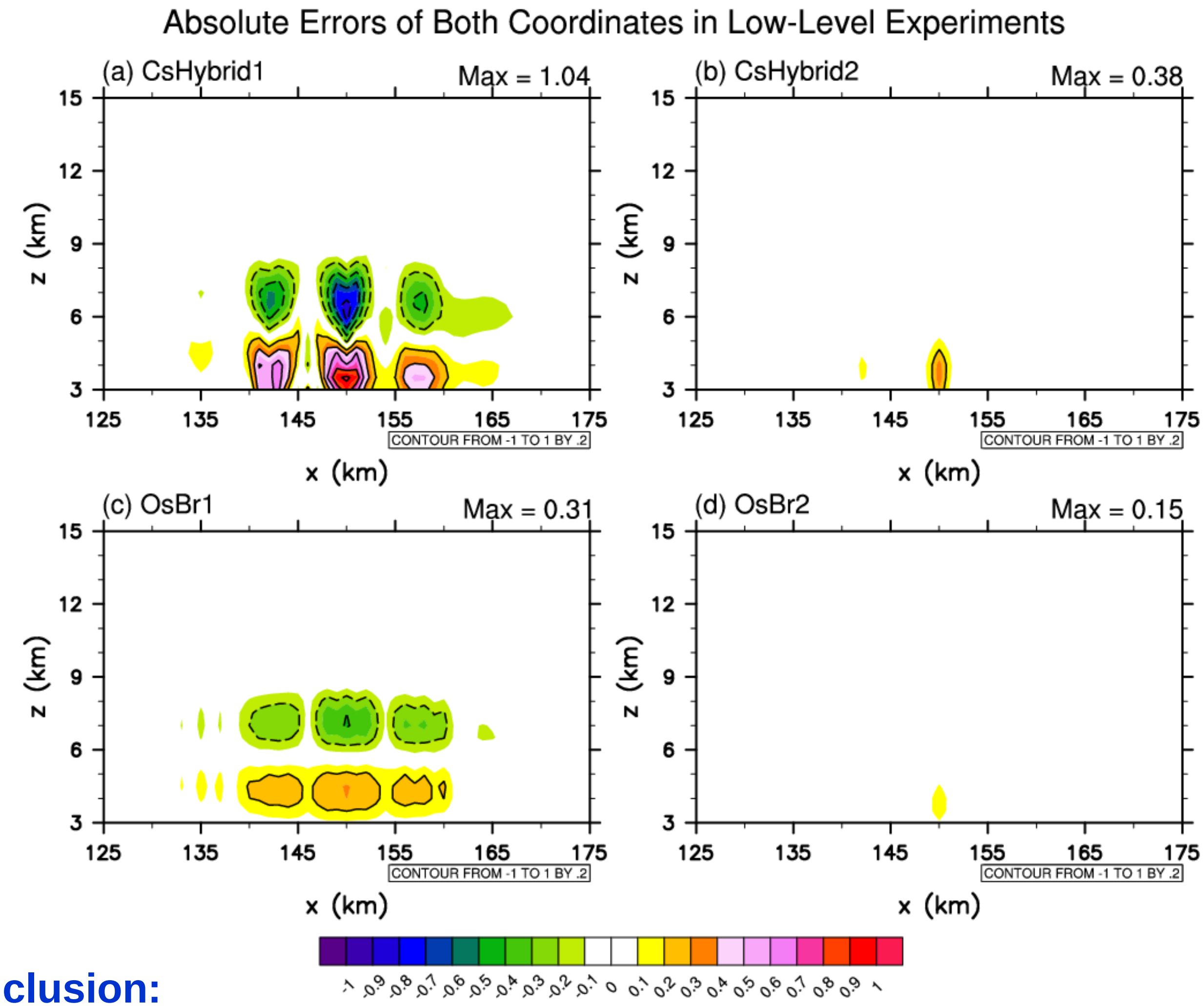
The advection at the beginning (*t* = 0), the middle (*t* = 200), and the end (*t* = 400) of the advection in the modified Schär-type (low-level) experiments.

Colored contours are the tracer *q*, with the contour interval of 0.1.

Conclusion:

The slower the decay speed of rotation parameter is, the more apparent the deformation is.

When the advection is over the top of the terrain, there is large deformation with different kind in both CsHybrid1 and OsBr1. At the end of the advection, the shape of the tracer in CsHybrid1 still has a large deformation; however, the shape of the tracer in OsBr1 is almost recovered.



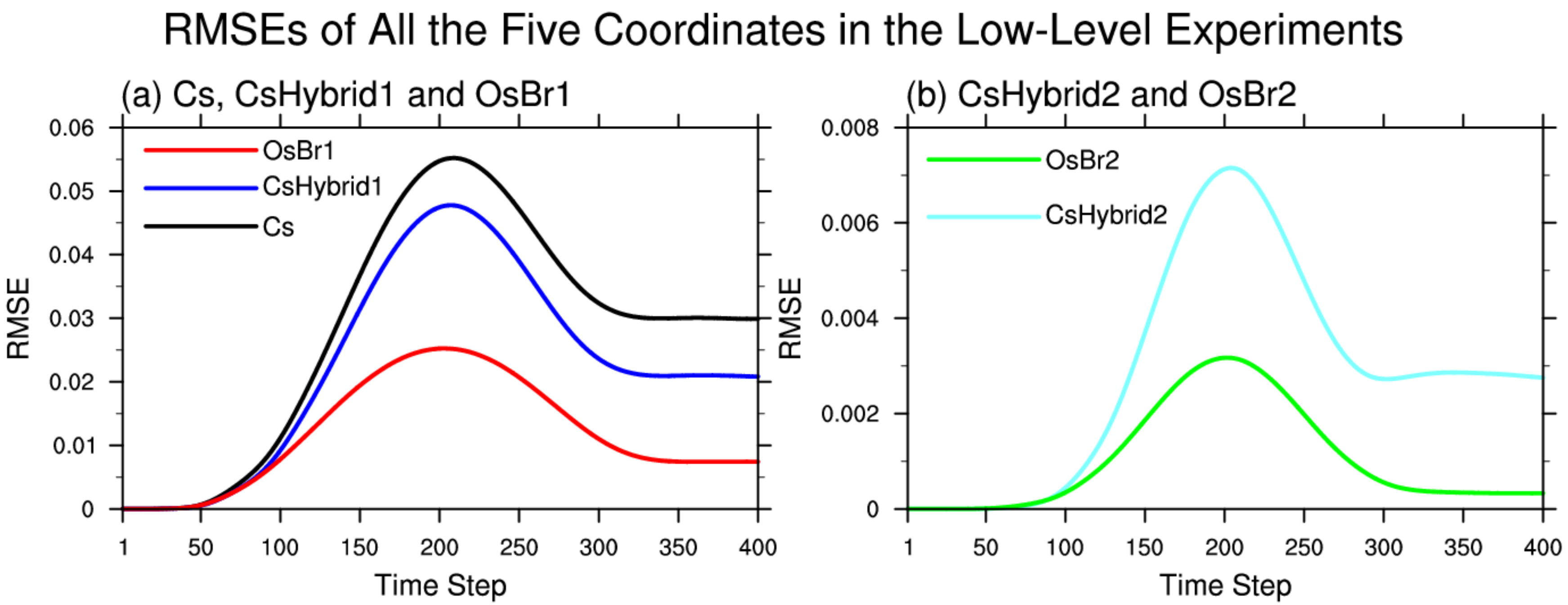
***t* = 200 s**

Absolute errors of the hybrid σ-coordinate and the OS-coordinate compared with the non-terrain simulation.

Shading represents the AE. The solid black contours are for positive values, and the dashed contours are for negative values. The contour interval is 0.2.

Conclusion:

The absolute errors in OS-coordinate are much smaller than those in the hybrid σ-coordinate.



RMSEs of all five experiments with respect to the non-terrain simulation in the modified Schär-type (low-level) experiments at every time step.

Conclusion:

The RMSEs reduction of the advection errors by the OS-coordinate is about 50% more compared with the corresponding hybrid σ-coordinate.

RMSE reduction by the OS-coordinate in the Schär-type (high-level) experiments

Experiments	***RMSEs***		***RMSEs reduction by the OS-coordinate Unit: %***	
	average	**maximum**	**average**	**maximum**
CsHybrid1	**0.015**	**0.035**	**26.9**	**30.5**
OsBr1	**0.011**	**0.024**		
CsHybrid2	**0.000068**	**0.00023**	**25.5**	**30.4**
OsBr2	**0.000051**	**0.00016**		

RMSE reduction by the OS-coordinate in the modified Schär (low-level) experiments

Experiments	***RMSEs***		***RMSEs reduction by the OS-coordinate Unit: %***	
	average	**maximum**	**average**	**maximum**
CsHybrid1	**0.029**	**0.048**	**47.5**	**47.2**
OsBr1	**0.0120**	**0.025**		
CsHybrid2	**0.0029**	**0.0072**	**63.5**	**55.7**
OsBr2	**0.0011**	**0.0032**		