Variable Order Mixed Finite Elements on Quadrilateral Grids for the Shallow Water Equations

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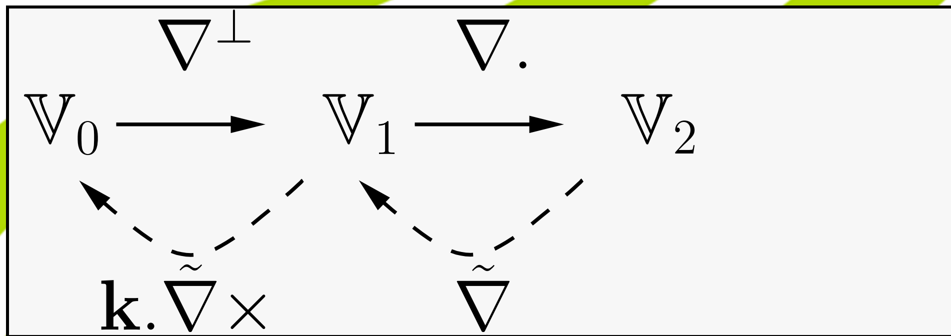
Overview

- Mixed finite element overview
- Raviart-Thomas based elements
- Dispersion analysis
- Transformation to the sphere
- Shallow water test results

Mixed Finite elements

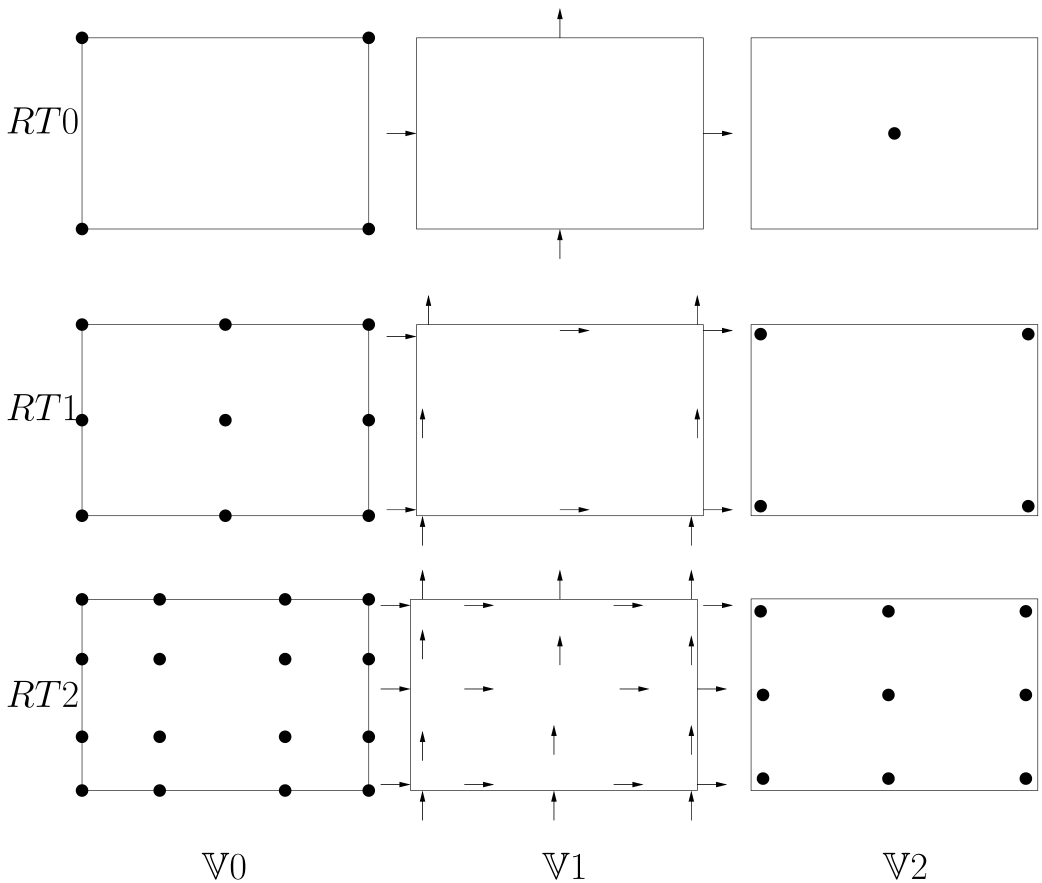
- For a 2D domain choose function spaces



- $\mathbb{V}_0$: Contains continuous scalar fuctions.
- $\mathbb{V}_1$: Contains vector fuctions with continuous normal components.
- $\mathbb{V}_2$: Contains discontinuous scalar fuctions.
- For quadrilateral elements the $Q^{k+1} - RT^k - Q^k_{DG}$ set of function spaces fit this complex.

Mixed Finite elements

- 2D linear shallow water equations:



Mixed Finite elements

- 2D linear shallow water equations:

$$\frac{\partial \Phi}{\partial t} + \Phi_0 \nabla . \mathbf{u} = 0,$$

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \Phi + f \mathbf{u}^{\perp} = 0$$

Mixed Finite elements

- 2D linear shallow water equations:
- becomes

$$\left\langle \rho, \frac{\partial \Phi}{\partial t} \right\rangle + \Phi_0 \langle \rho, \nabla.\mathbf{u} \rangle = 0,$$

$$\left\langle \mathbf{w}, \frac{\partial \mathbf{u}}{\partial t} \right\rangle + \left\langle \mathbf{w}, \tilde{\nabla}\Phi \right\rangle + f \left\langle \mathbf{w}, \mathbf{u}^{\perp} \right\rangle = 0$$

$$\langle a, b \rangle = \int_{\Omega} ab d\mathbf{x}$$

Mixed Finite elements

- 2D linear shallow water equations:
- becomes

$$M_\Phi \frac{\partial}{\partial t}\hat{\Phi} + \Phi_0 D\hat{u} = 0$$

$$M_u \frac{\partial}{\partial t}\hat{u} + \ D^T\hat{\Phi} + fC\hat{u} = 0$$

- $M_\phi^{i,j} = \left\langle \rho^i, \rho^j \right\rangle$, $D^{i,j} = \left\langle \rho^i, \nabla.\mathbf{w}^j \right\rangle$.
- $M_u^{i,j} = \left\langle \mathbf{w}^i, \mathbf{w}^j \right\rangle$, $C^{i,j} = \left\langle \mathbf{w}^i, \left(\mathbf{w}^j\right)^\perp \right\rangle$.
- Increasing order increases the number of branches of the dispersion relation

Mixed Finite elements

- 2D linear shallow water equations:
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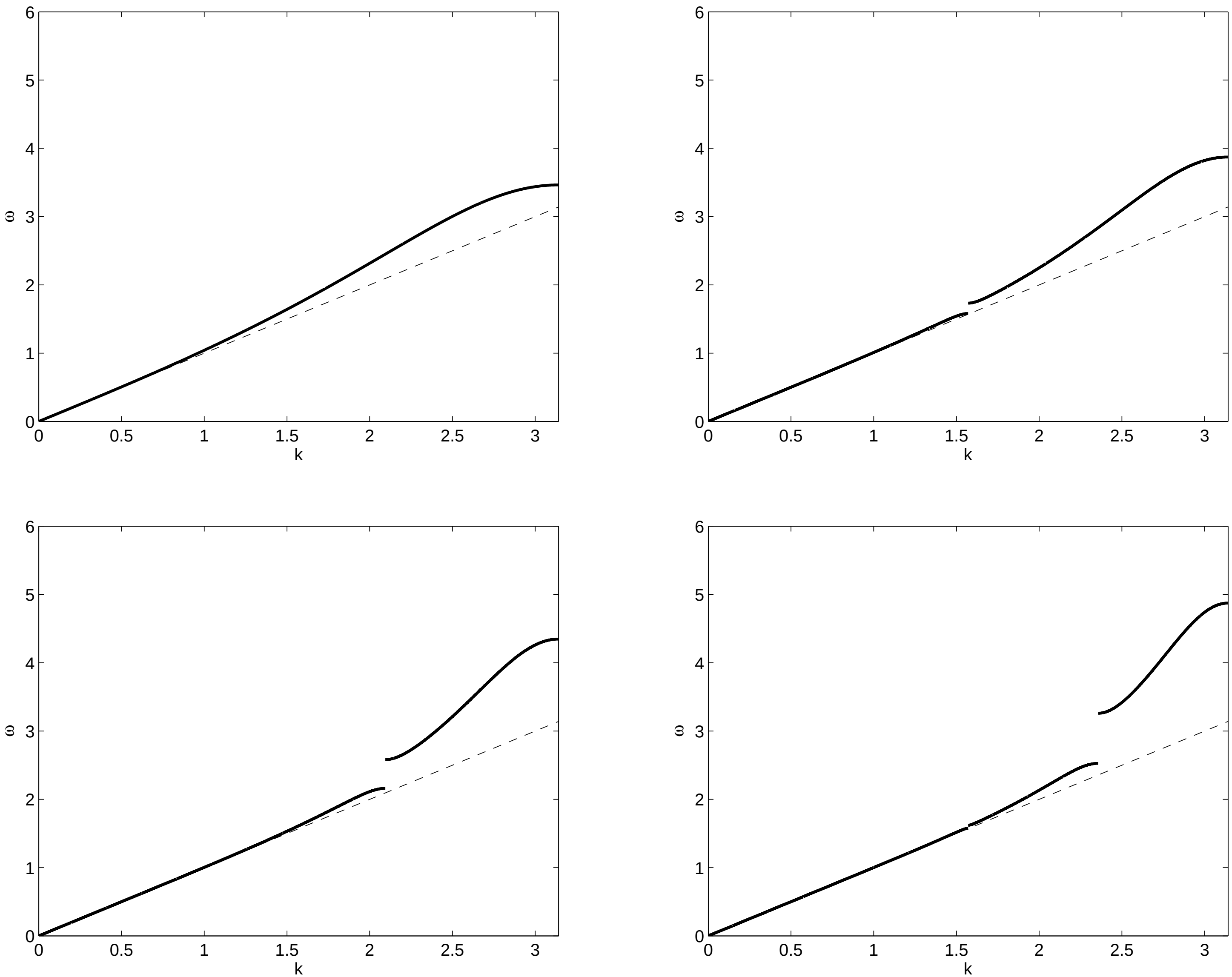
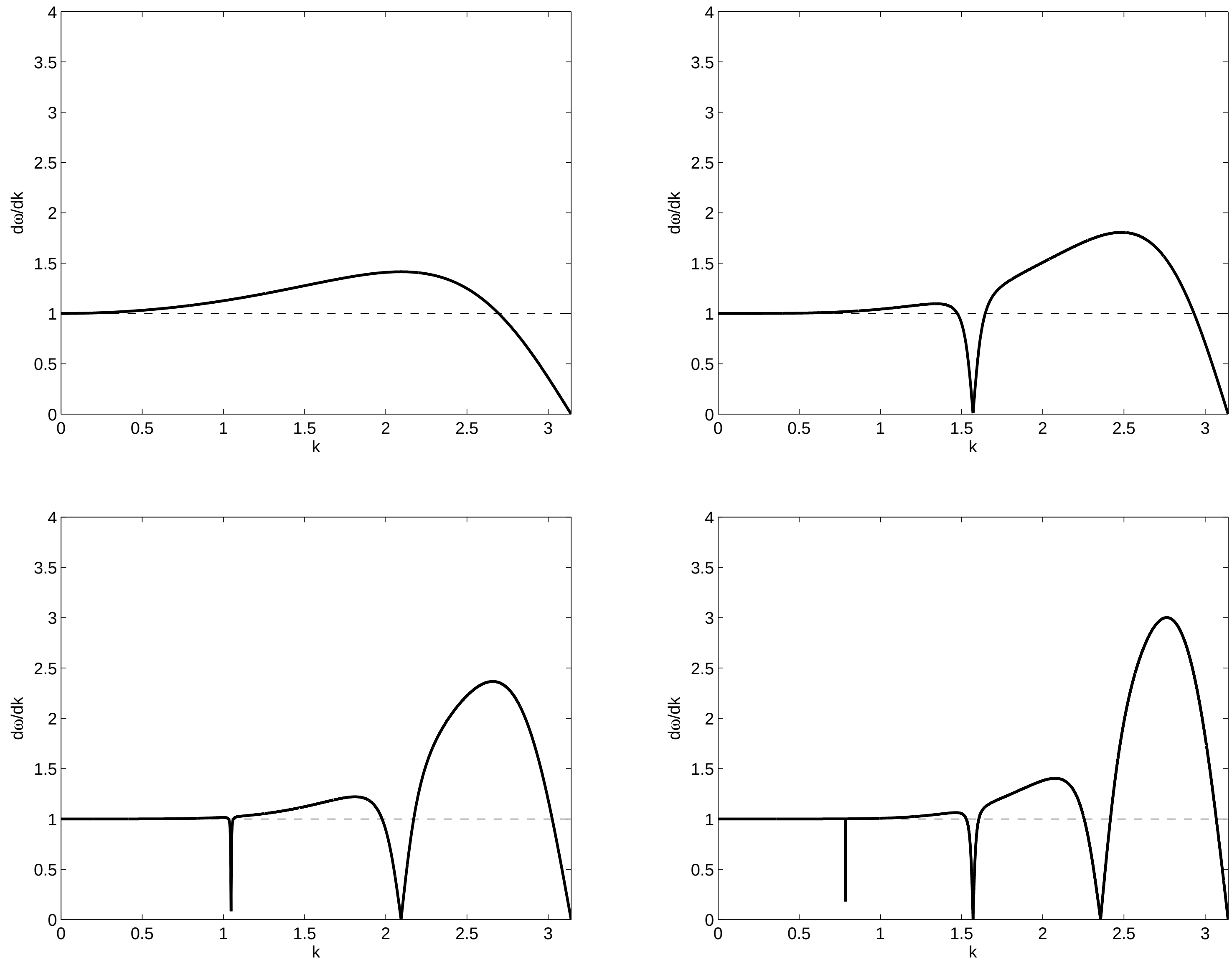
- Increasing order increases the number of branches of the dispersion relation
- To study dispersion properties of system:

$$\hat{\Phi} = P\exp\left[i\left(\mathbf{k}.\mathbf{x}_n - \omega t\right)\right], \quad \hat{u} = U\exp\left[i\left(\mathbf{k}.\mathbf{x}_n - \omega t\right)\right]$$

Dispersion Properties - Gravity Waves



Dispersion Properties - Gravity Waves



Partial Mass Lumping

- For RT1 elements in 1D:

$$M = \frac{1}{60}\begin{bmatrix} 8 & 4 & -2 \\ 4 & 32 & 4 \\ -2 & 4 & 8 \end{bmatrix} \Longrightarrow \frac{1}{60}\begin{bmatrix} 8+\alpha & 4 & -2-\alpha \\ 4 & 32 & 4 \\ -2-\alpha & 4 & 8+\alpha \end{bmatrix}$$

- $\alpha = 1/20$ removes discontinuity

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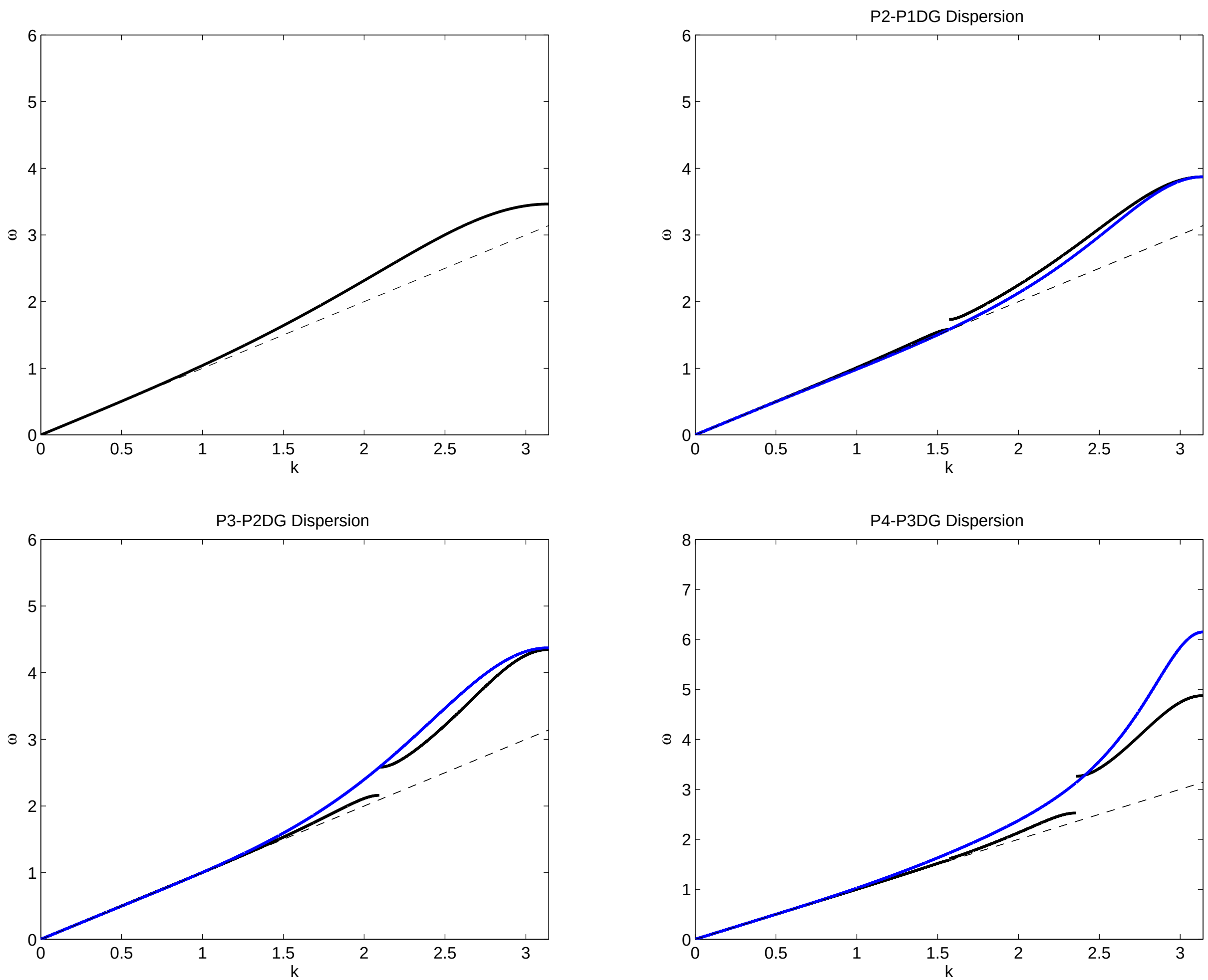
- $\alpha = 1/20$ removes discontinuity
- Rewrite as $M \longrightarrow M + \widetilde{M}$

$$M = \frac{1}{60}\begin{bmatrix} 8 & 4 & -2 \\ 4 & 32 & 4 \\ -2 & 4 & 8 \end{bmatrix} + \frac{1}{60}\begin{bmatrix} \alpha & 0 & -\alpha \\ 0 & 0 & 0 \\ -\alpha & 0 & \alpha \end{bmatrix}$$

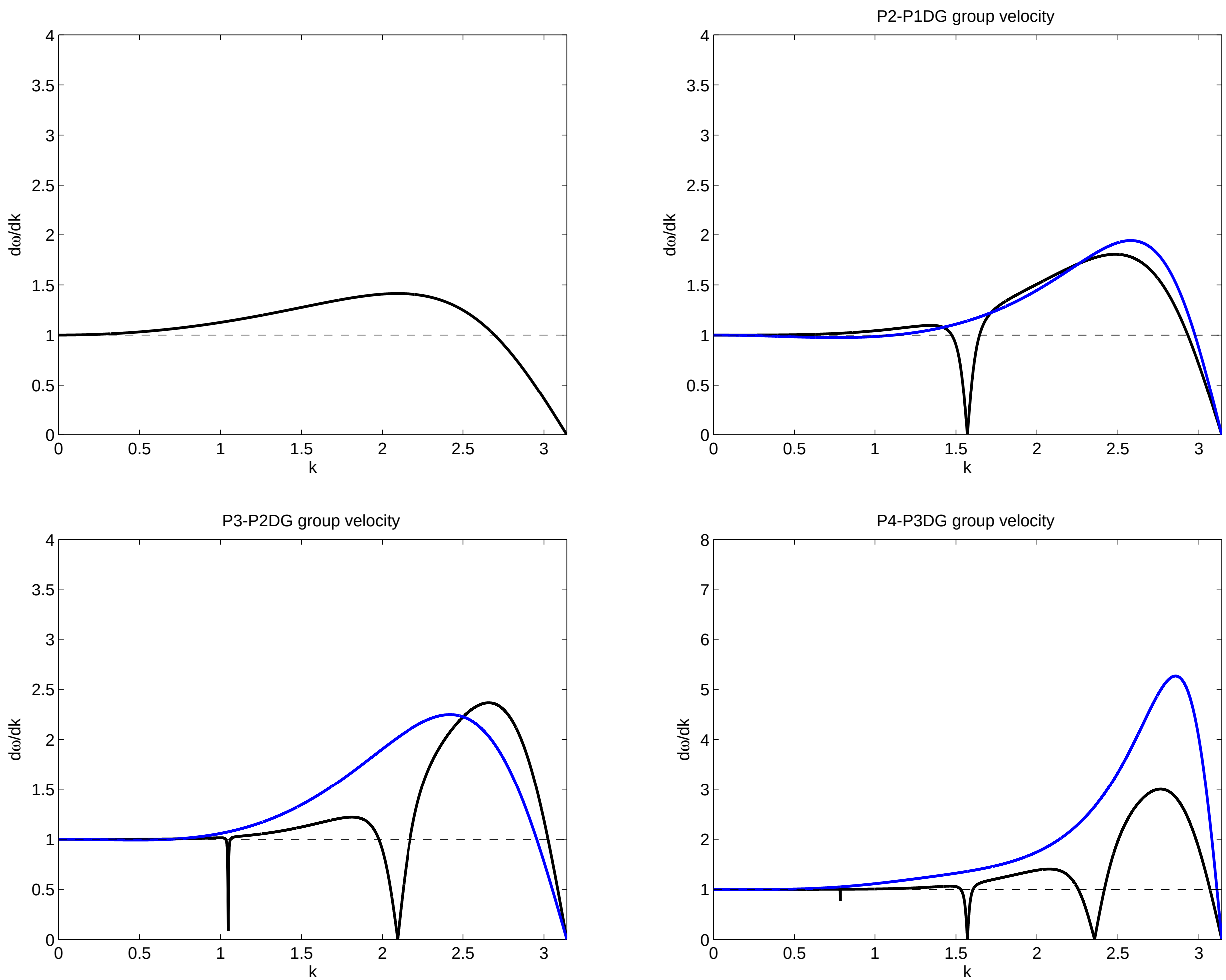
Dispersion Properties - Gravity Waves

Modifying M_u allows discontinuities to be removed

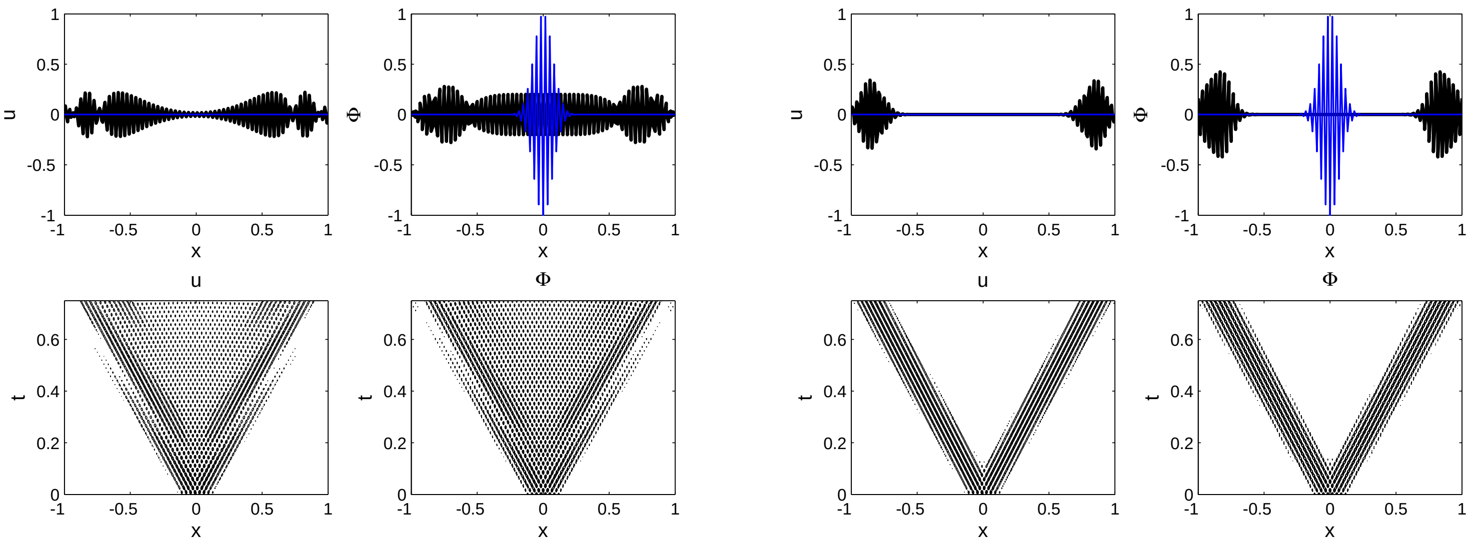


Dispersion Properties - Gravity Waves

Modifying M_u allows discontinuities to be removed



Dispersion Properties - Mixed FEM



Extension to the sphere

- Map from uniform reference element to element on the sphere using $\phi : \hat{\Omega} \rightarrow \Omega$
- Use Contravariant Piola transform for vector fields (maintains normal flow)

$$\mathbf{u} = \frac{J\hat{\mathbf{u}}}{\det J}, \qquad J = \frac{\partial \phi \left(\mathbf{x}\right)}{\partial \boldsymbol{\xi}}$$

- Nonlinear SWE:

$$h_t + \nabla . \mathbf{F} = 0,$$

$$\mathbf{u}_t + q\mathbf{F}^{\perp} + \nabla \left[g\left(h + h_0\right) + \frac{1}{2}\mathbf{u}.\mathbf{u} \right] = 0,$$

Extension to the sphere

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$$\mathbf{u} = \frac{J\hat{\mathbf{u}}}{\det J}, \qquad J = \frac{\partial \phi(\mathbf{x})}{\partial \boldsymbol{\xi}}$$

- Nonlinear SWE:

$$\left\langle \rho, \frac{1}{\det J} h_t \right\rangle + \langle \rho, \nabla . \mathbf{F} \rangle = 0,$$

$$\left\langle J\mathbf{w}, \frac{J}{\det J} \mathbf{u}_t \right\rangle + \langle \mathbf{w}, q\mathbf{F}^{\perp} \rangle - \left\langle \nabla . \mathbf{w}, \left[g(h + h_0) + \frac{1}{2} \left| \frac{J\mathbf{u}}{\det J} \right|^2 \right] \right\rangle = 0,$$

Partial Mass Lumping

In general mapping is non affine $\rightarrow \det J \neq const$

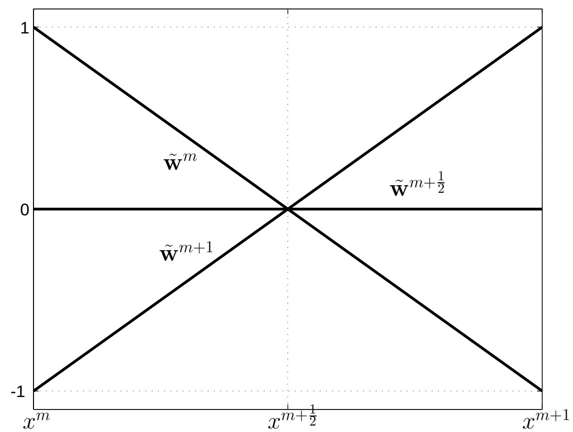
- Mass Matrix entries become:

$$M^{i,j} = \int \frac{J\mathbf{w}^i}{\det(J)} \cdot J\mathbf{w}^j dA$$

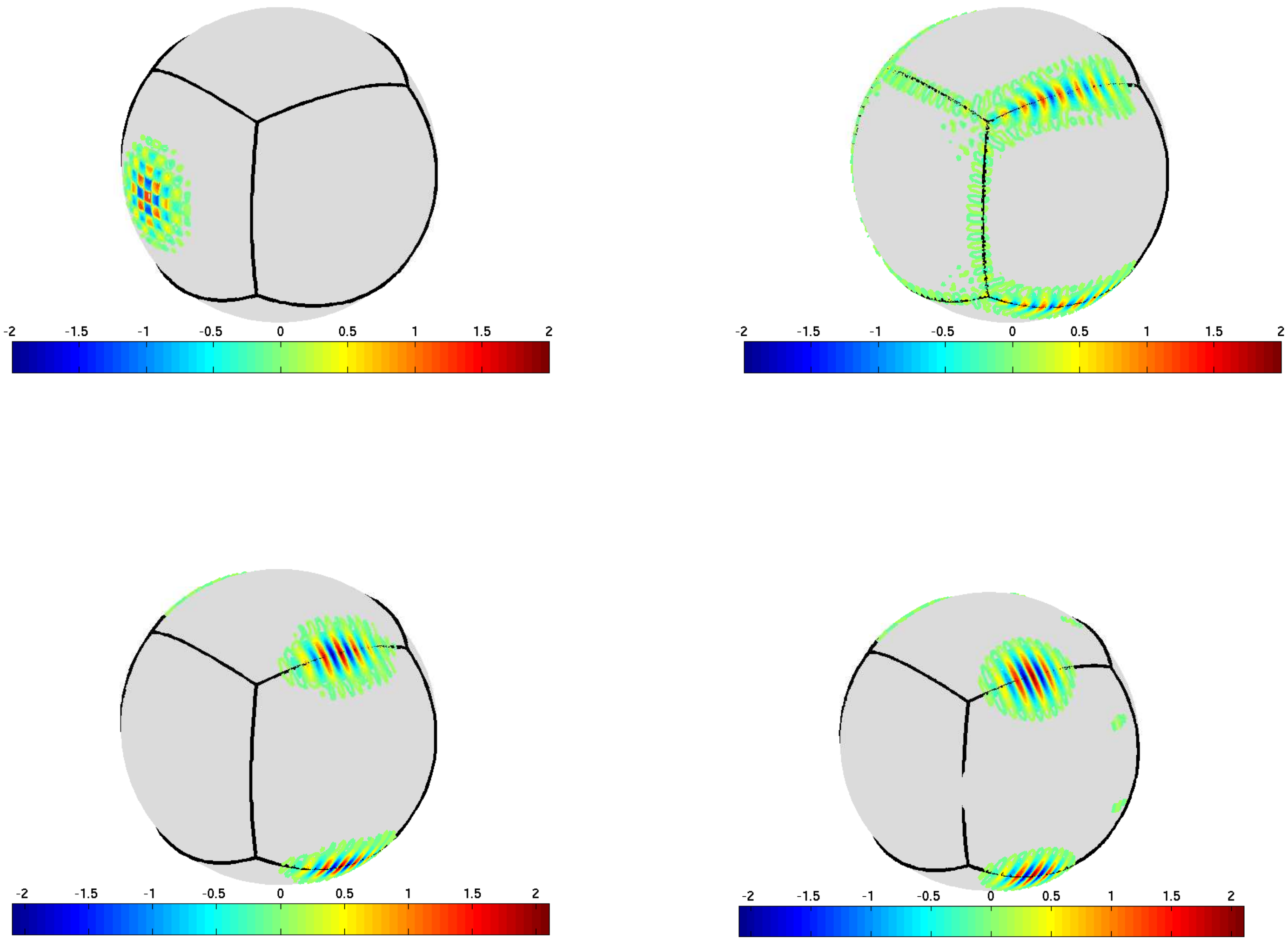
- This cannot be integrated exactly: How to apply mass lumping?
- Compute $\widetilde{M}$ as

$$\widetilde{M}^{i,j} = \int \frac{J\tilde{\mathbf{w}}^i}{\det(J)} \cdot J\tilde{\mathbf{w}}^j dA$$

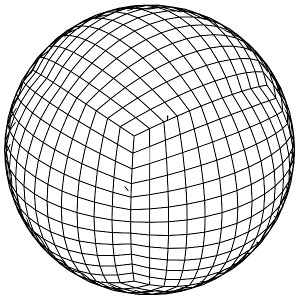
$\tilde{\mathbf{w}}$ are some simple basis functions

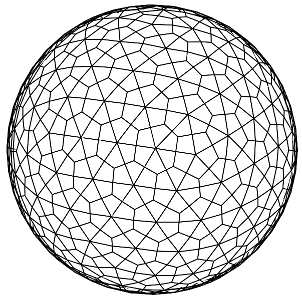


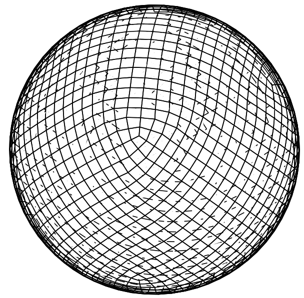
Dispersion Properties - Spherical

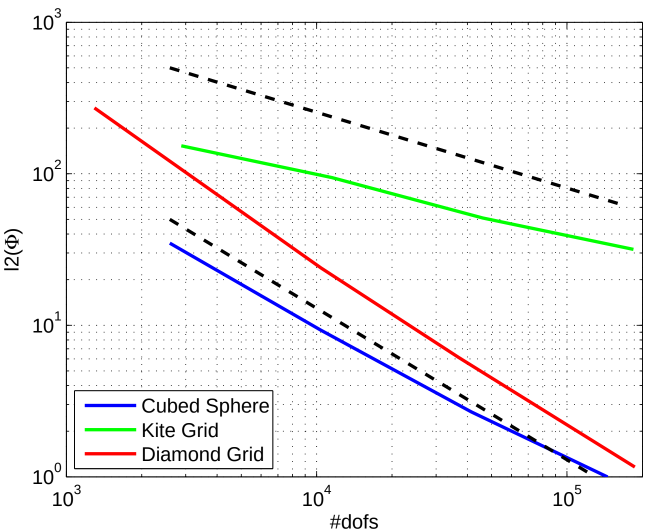


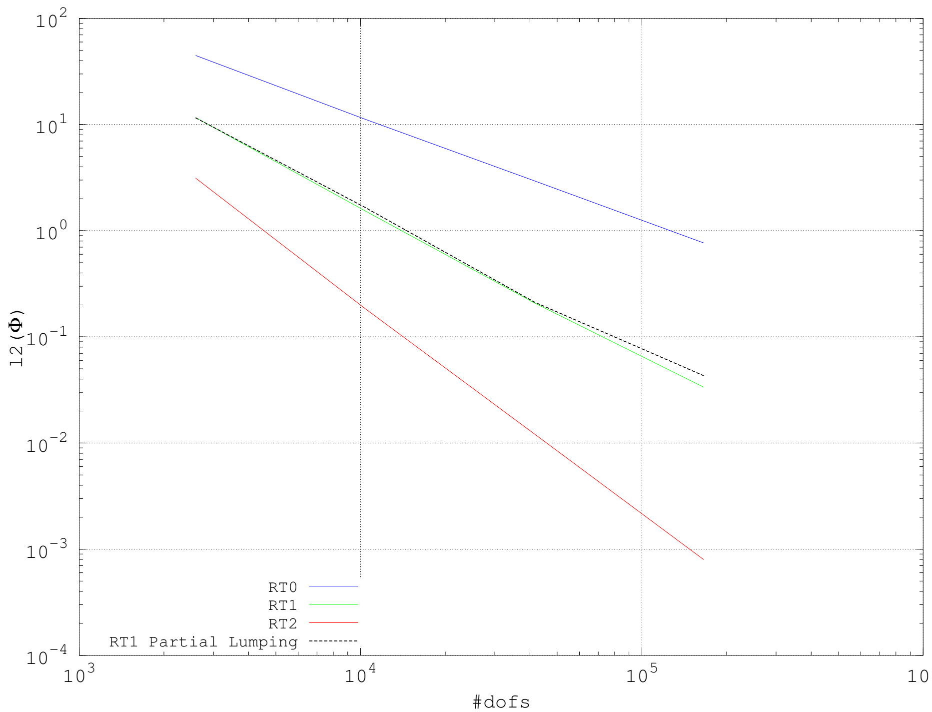
SWE Results - solid body rotation





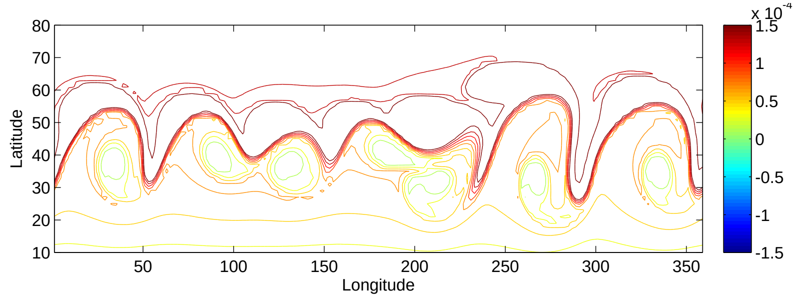




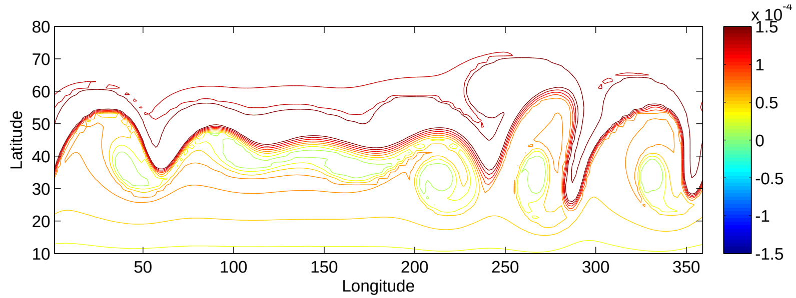


SWE Results - Galewsky Test - Cubed Sphere

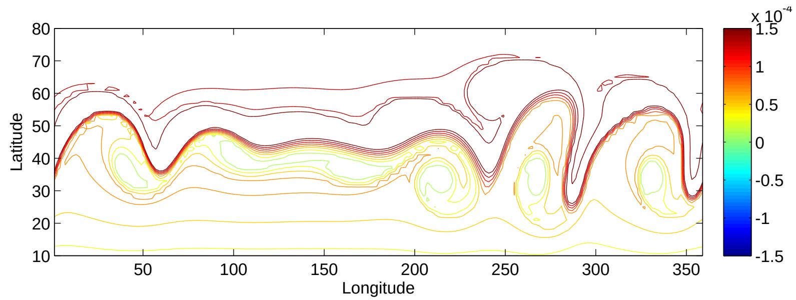
RT0 - $663552 dofs$



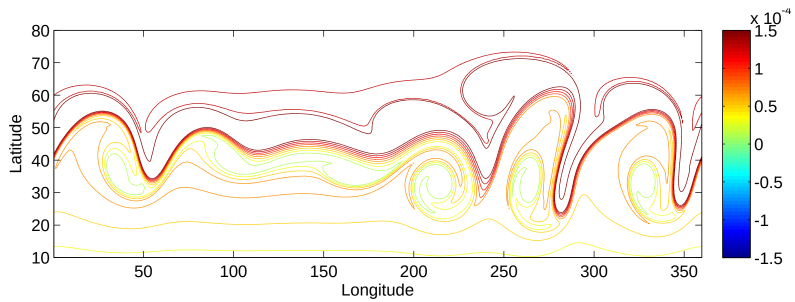
RT1 - $663552 dofs$



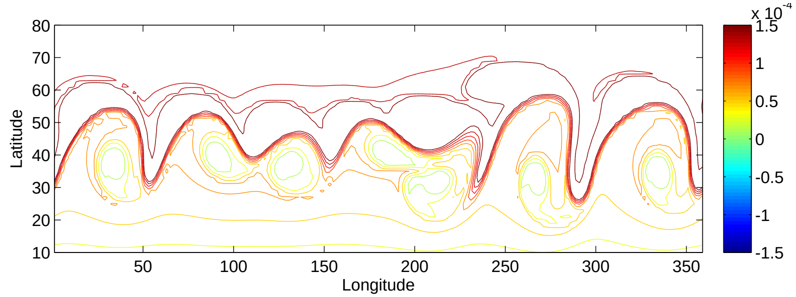
RT2 - $663552 dofs$



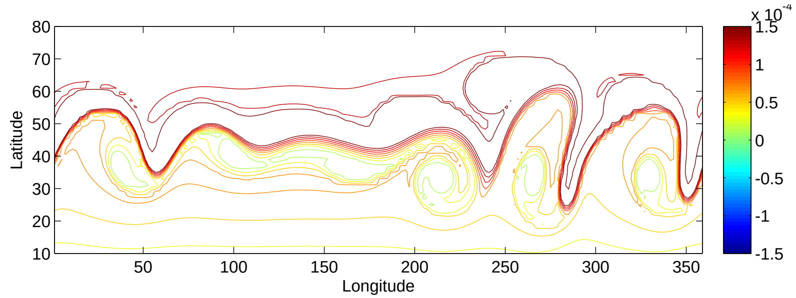
EG - $614400 dofs$



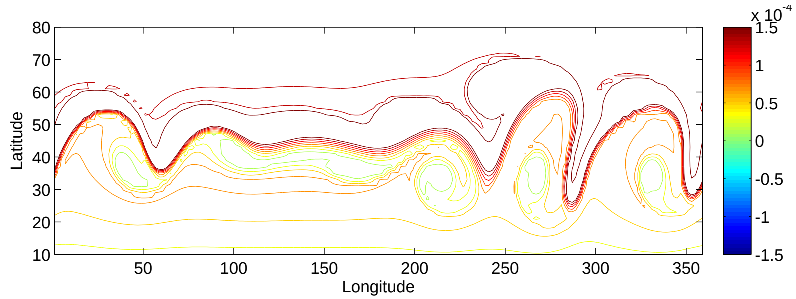
SWE Results - Galewsky Test - Cubed Sphere



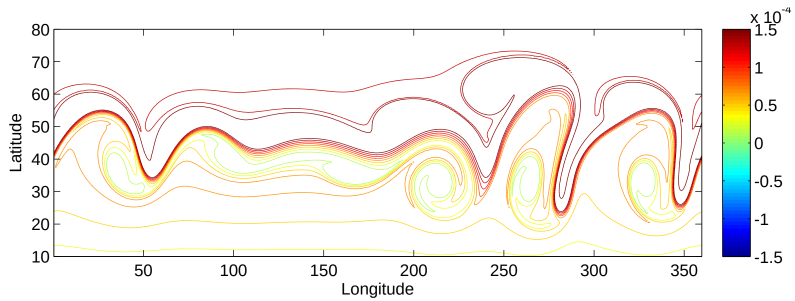
RT0 - $663552 dofs$



RT1 - Mass lump



RT2 - $663552 dofs$

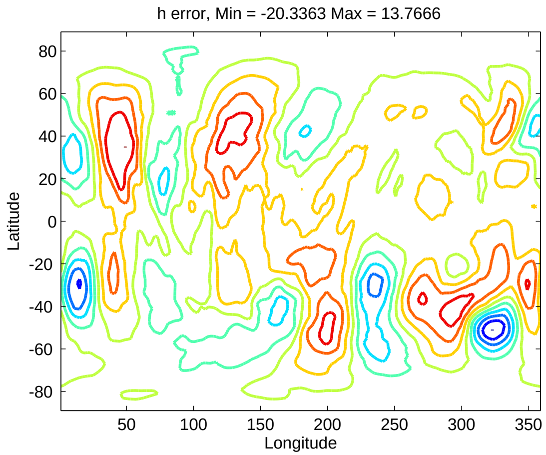


EG - $614400 dofs$

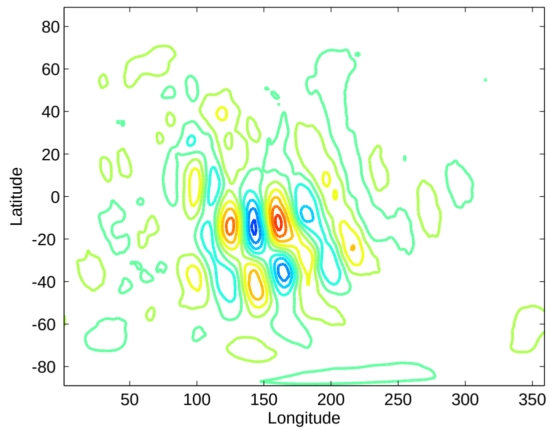
Conclusions

- Mixed Element Methods have a large number of desirable properties
- Above lowest order, a number of stationary modes exist
- These can in theory be removed via partial mass lumping
- Stationary modes do not appear to effect results
- Partial mass lumping does not greatly effect results
- Next steps:
  - Forced shallow water test
  - 2D vertical slice and 3D prototype models

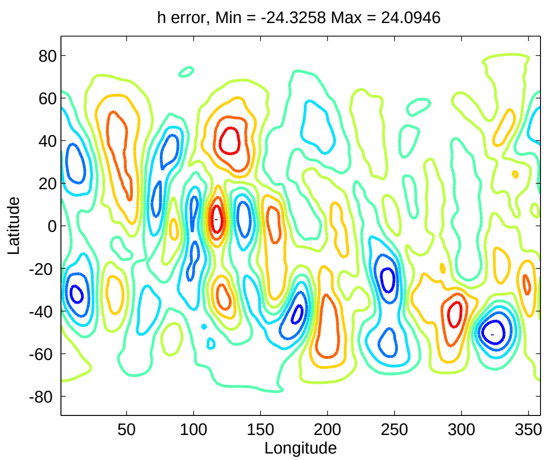
SWE Results - Flow over a mountain



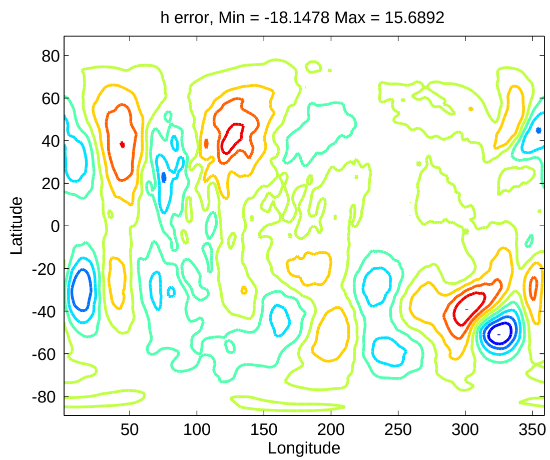
(a) RT1 error



(b) RT1 - Difference

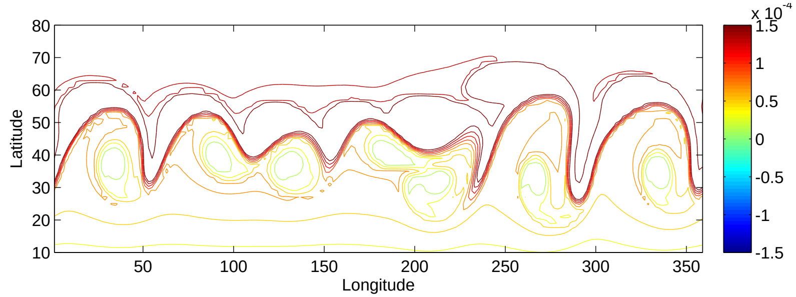


(c) RT0 error

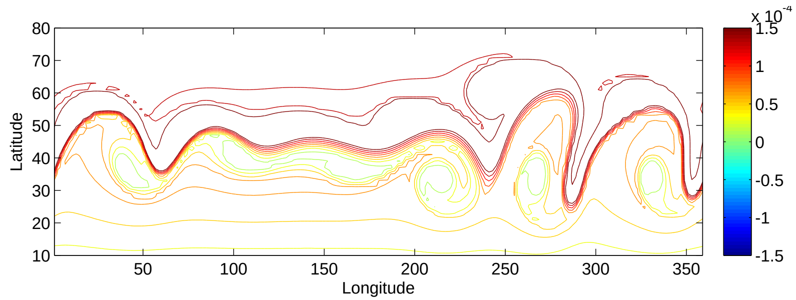


(d) RT2 error

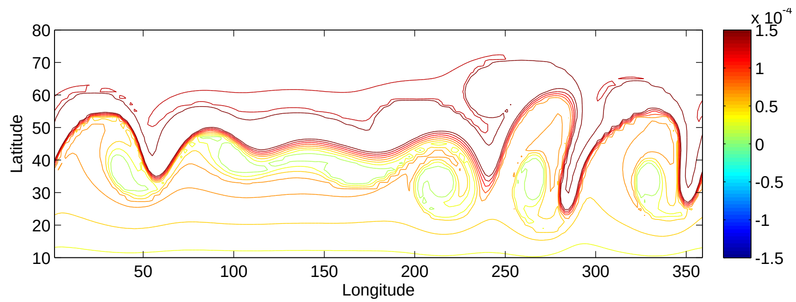
SWE Results - Galewsky Test - Cubed Sphere



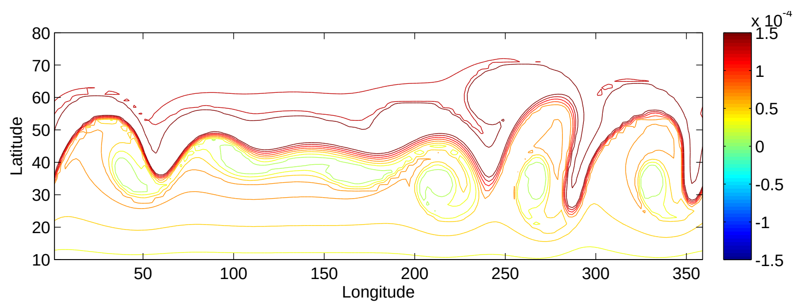
RT0 - $663552dofs$



RT1 - $663552dofs$

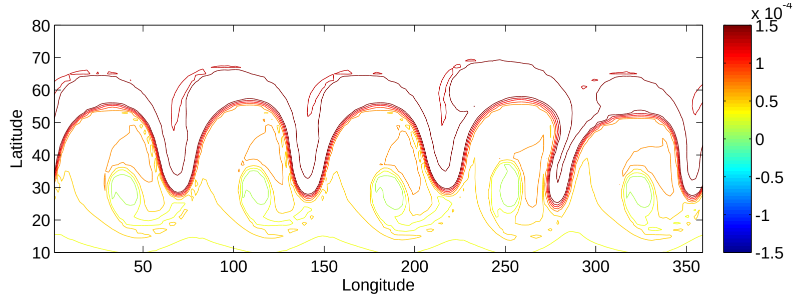


RT1 - Mass lump

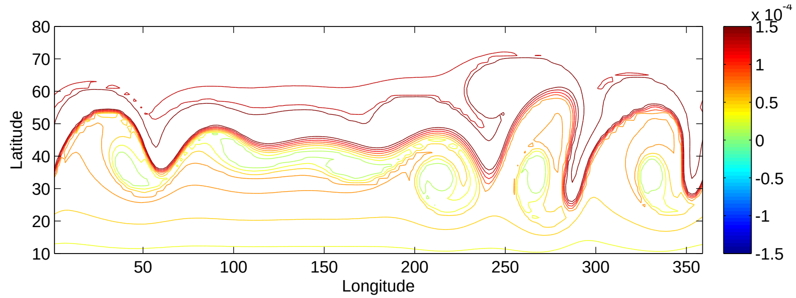


RT2 - $663552dofs$

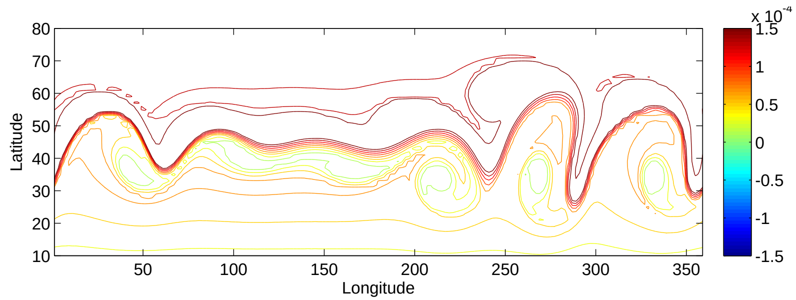
SWE Results - Galewsky Test - Kite Grid RT0



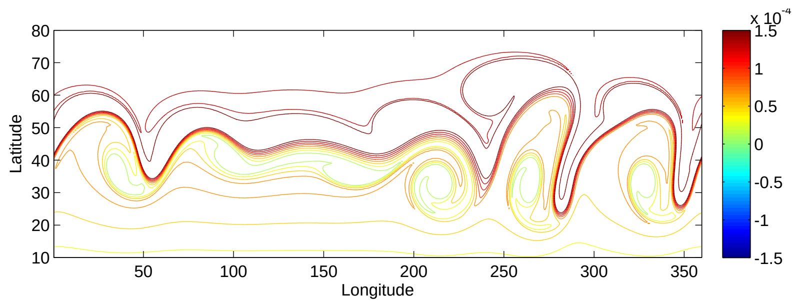
RT0 - $737280 dofs$



RT1 - $737280 dofs$



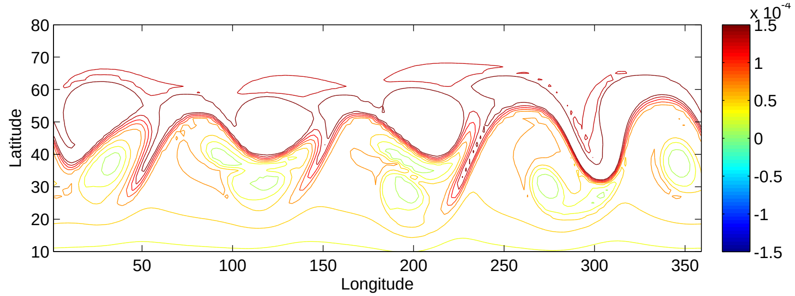
RT2 - $414720 dofs$



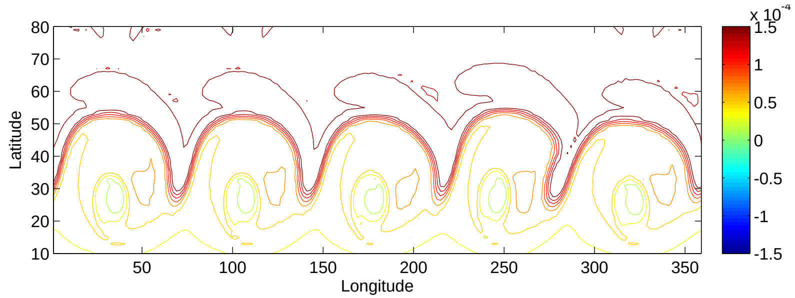
EG - $614400 dofs$

SWE Results - Galewsky Test -

Cubed Sphere - $165888 dofs$



Kite Grid - $184320 dofs$



Diamond Grid - $147456 dofs$