Controlling the energy spectrum

Keith Myerscough
CWI, Amsterdam

PDEs on the Sphere, April 2014

Netherlands Organisation for Scientific Research

Centrum Wiskunde & Informatica

with Jason Frank & Ben Leimkuhler

Atmospheric energy spectrum

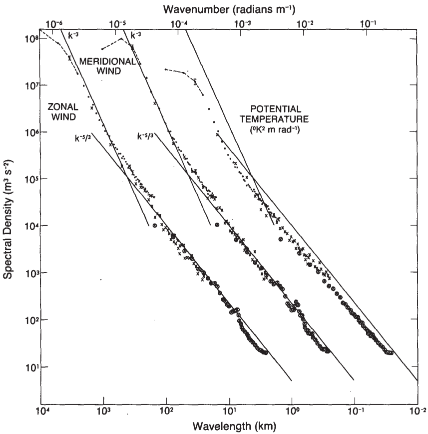


FIG. 3. Variance power spectra of wind and potential temperature near the tropopause from GASP aircraft data. The spectra for meridional wind and temperature are shifted one and two decades to the right, respectively; lines with slopes −3 and −⁵⁄₃ are entered at the same relative coordinates for each variable for comparison.

Incompressible Navier-Stokes

In a 2D periodic box:

$$\omega_t + J(\psi, \omega) = f + \nu \Delta \omega - \alpha \omega,$$

where the vorticity $\omega = \Delta \psi$ and

$$J(\psi, \omega) = \psi_x \omega_y - \psi_y \omega_x.$$

The friction $-\alpha\omega$ is restricted to the largest scales

Pseudo-spectral method

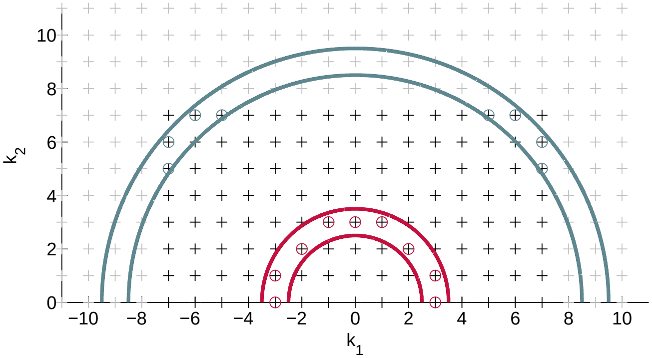
$$\omega_{\mathbf{k}}(t) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \omega(\mathbf{x}, t) e^{-i\mathbf{k}\cdot\mathbf{x}} \, \mathrm{d}\mathbf{x}$$

with the dynamics

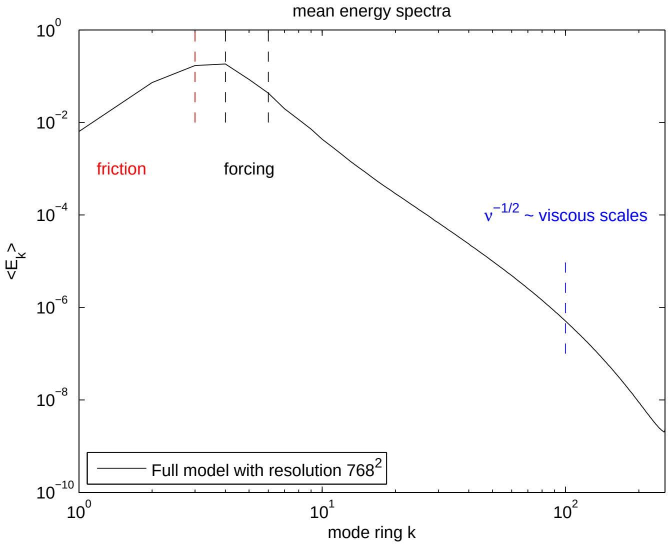
$$\dot{\omega}_{\mathbf{k}} + J_{\mathbf{k}}(\boldsymbol{\omega}) = f_{\mathbf{k}} + \nu \Delta_{\mathbf{k}} \omega_{\mathbf{k}} - \alpha \omega_{\mathbf{k}} \mathbf{1}_{|\mathbf{k}| \leq 3},$$

Energy spectrum

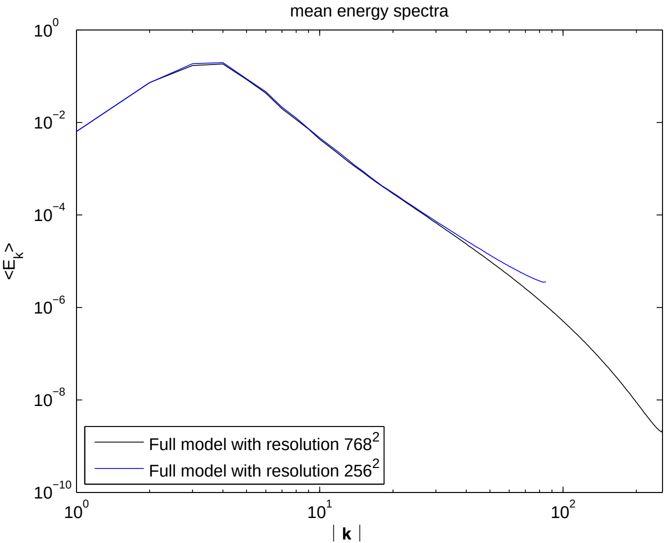
$$E_k = \frac{-1}{2} \sum_{k-1/2<|\mathbf{k}|<k+1/2} \Delta_{\mathbf{k}}^{-1} \omega_{\mathbf{k}} \omega_{\mathbf{k}}^{*} \tag{1}$$



Two-dimensional turbulent energy spectrum

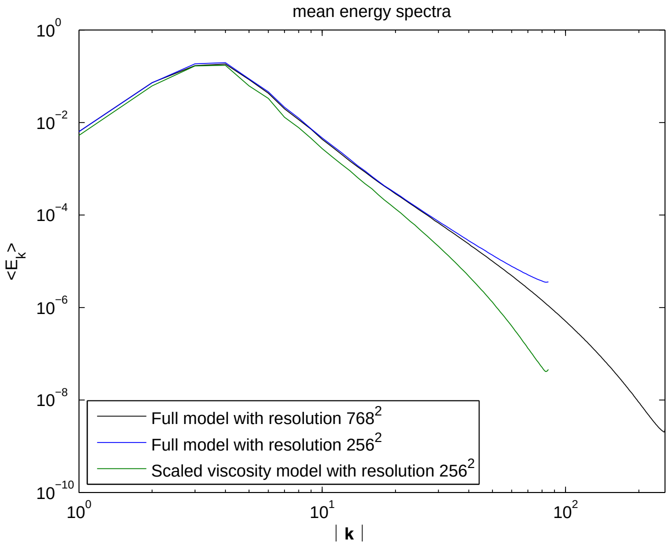


Truncated energy spectrum



Artificial viscosity

Increase ν such that the viscosity acts at a resolved scales.

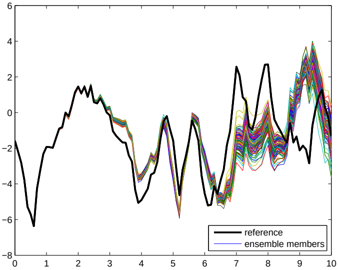


Dispersivity

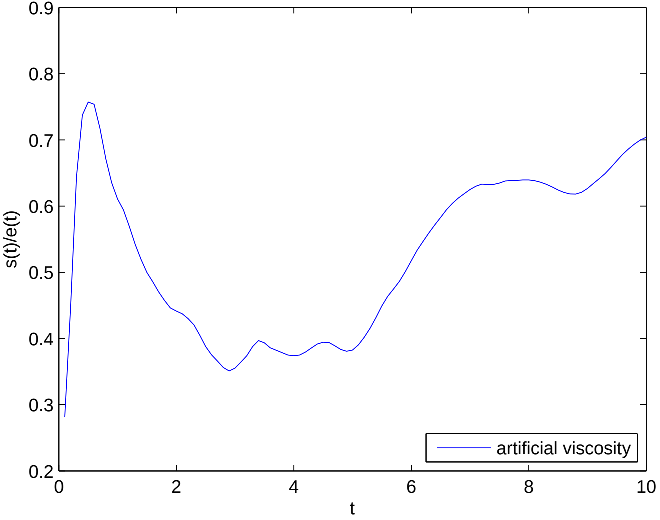
Compare spread of ensemble members to RMS error of ensemble members.

$$s(t) = \left\langle |a_i - \bar{a}|^2 \right\rangle_i , \quad e(t) = \left\langle |a_i - A|^2 \right\rangle_i$$

a_i is ensemble observable, $\bar{a}$ the ensemble mean, A the "truth".

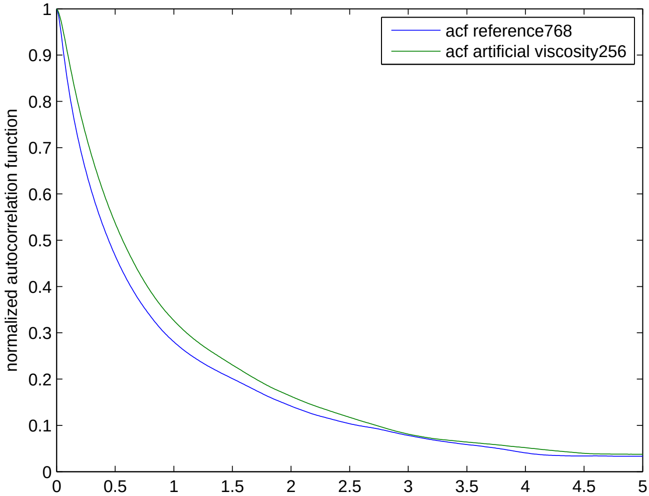


Dispersivity



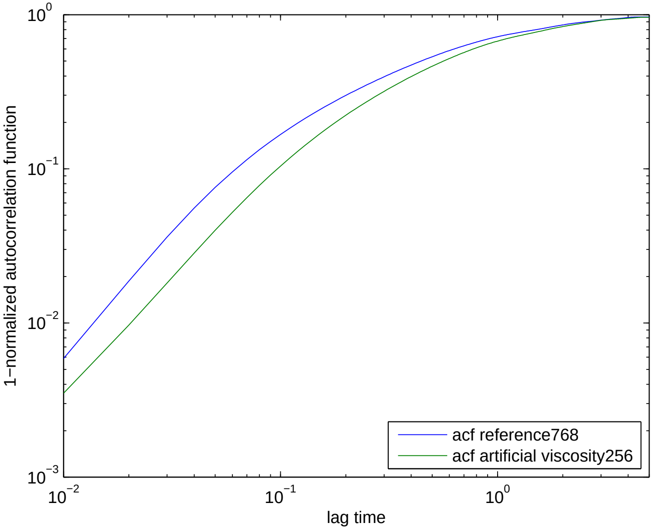
Auto-correlation functions

$$R_{\omega\omega}(\tau) = \frac{1}{T}\int_0^T \omega(t+\tau)\omega(t)\,\mathrm{d}t$$



Auto-correlation functions

$$1 - R_{\omega\omega}(\tau)$$



Truncation drawbacks

- Energy spectrum does not match the observed data
- Insufficient energy at small scales causes
  - underdispersive ensemble
  - overly time-correlated solutions
- Well-studied problem
  - Backscatter algorithms (Shutts, Berner et al.)
  - Stochastic subgrid models (Berloff, Marstorp et al, Crommelin & Vanden-Eijnden)
  - Cut-off filters (Tullock & Smith 2009)
  - Hyperviscosity (textbook)

Correction device

- Could *impose* the spectrum with constraints

Correction device

- Could *impose* the spectrum with constraints

$$\dot{\omega}_{\mathbf{k}} + J_{\mathbf{k}}(\boldsymbol{\omega}) = f_{\mathbf{k}} + \nu\Delta_{\mathbf{k}}\omega_{\mathbf{k}} - \alpha\omega_{\mathbf{k}}\mathbf{1}_{|\mathbf{k}|\leq 3} - \sum_{l}\xi_l\partial_{\omega_{\mathbf{k}}}c_l(\boldsymbol{\omega})$$
$$0 = c_l(\boldsymbol{\omega}) - C_l.$$

  A Differential Algebraic Equation with Lagrange multipliers ξ_l
  $c_l(\boldsymbol{\omega}) = E_l(\boldsymbol{\omega}), \quad C_l = \langle E_l\rangle_{\text{data}}$
- Too strict!

Correction device

- Only the average has to be controlled
- Similar to a *thermostat* in Molecular Dynamics

Correction device

- Only the average has to be controlled
- Similar to a *thermostat* in Molecular Dynamics
  - Nosé-Hoover thermostat

$$
\begin{aligned}
\dot{q} &= p \\
\dot{p} &= -\nabla V(q) - \xi p \\
\mu\dot{\xi} &= K(p) - nk_B T
\end{aligned}
$$

Correction device

- Only the average has to be controlled
- Similar to a *thermostat* in Molecular Dynamics
  - Nosé-Hoover thermostat

$$\begin{aligned} \dot{q} &= p \\ \dot{p} &= -\nabla V(q) - \xi p \\ \mu\dot{\xi} &= K(p) - nk_B T \end{aligned}$$

  - Leimkuhler et al. ('08)

$$\begin{aligned} \dot{q} &= p \\ \dot{p} &= -\nabla V(q) - \xi p \\ \mu\xi &= \frac{1}{\int_0^t \phi(s)\,\mathrm{d}s} \int_0^t \phi(t-s) K(p(s))\,\mathrm{d}s - nk_B T \end{aligned}$$

Correction device

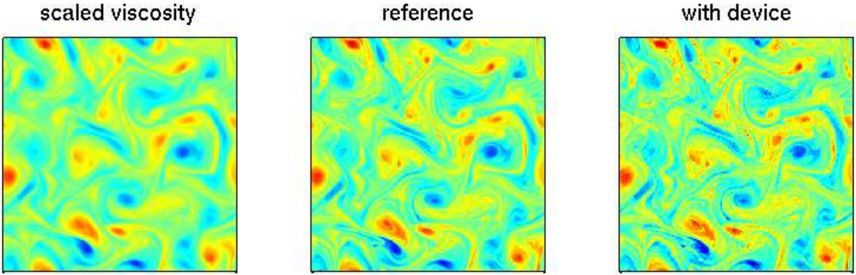
- Combine previous ideas

$$\dot{\omega}_{\mathbf{k}} + J_{\mathbf{k}}(\boldsymbol{\omega}) = f_{\mathbf{k}} + \nu \Delta_{\mathbf{k}} \omega_{\mathbf{k}} - \alpha \omega_{\mathbf{k}} \mathbf{1}_{|\mathbf{k}| \leq 3} - \sum_{l} \xi_l \partial_{\omega_{\mathbf{k}}} c_l(\boldsymbol{\omega})$$

$$\mu \dot{\xi}_l = \frac{1}{\int_0^t \phi(s)\,\mathrm{d}s} \int_0^t \phi(t-s) c_l\left(\boldsymbol{\omega}(s)\right) \mathrm{d}s - C_l.$$

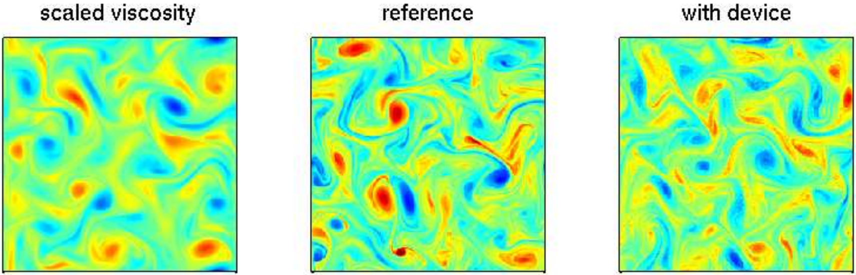
Vorticity field snapshot

After t=1

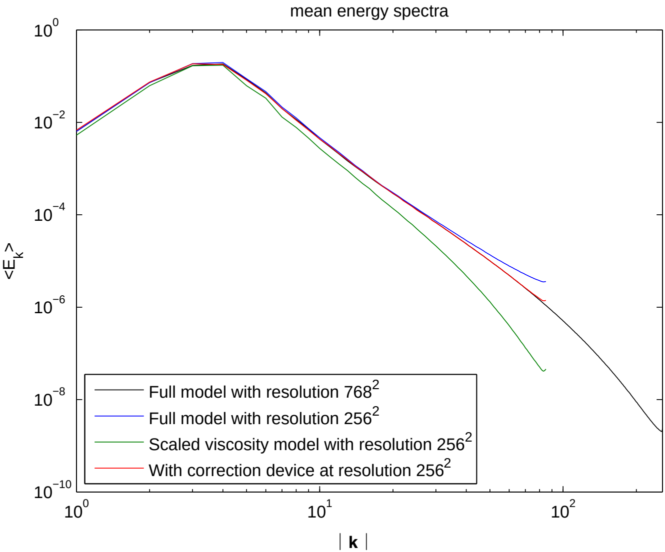


Vorticity field snapshot

After t=10

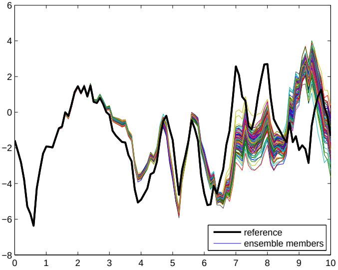


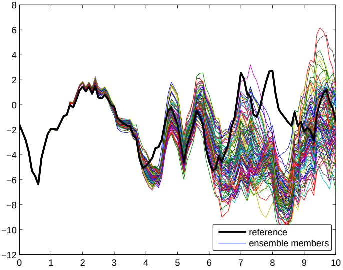
Spectrum



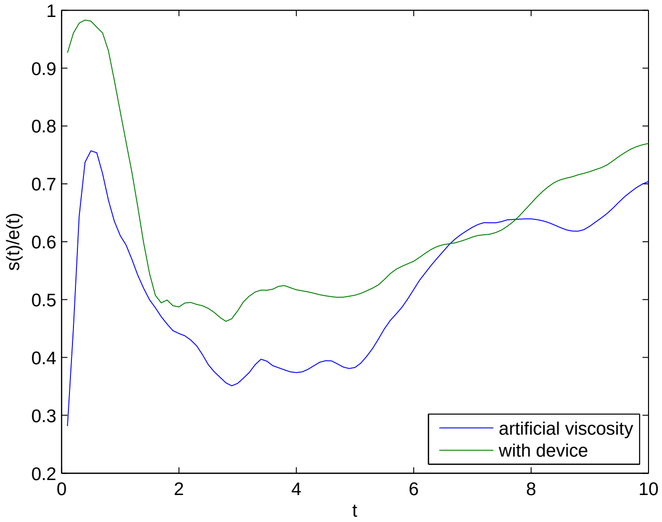
mean energy spectra
10^0
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
$<E_k>$
10^0
10^1
10^2
$|\mathbf{k}|$
Full model with resolution 768^2
Full model with resolution 256^2
Scaled viscosity model with resolution 256^2
With correction device at resolution 256^2

Dispersivity



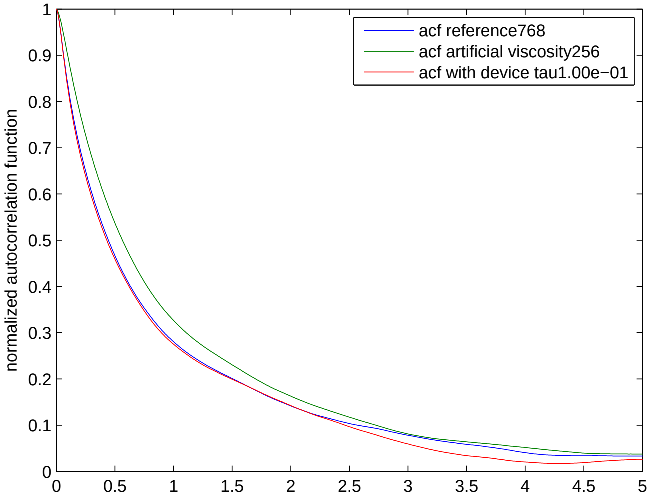


Dispersivity



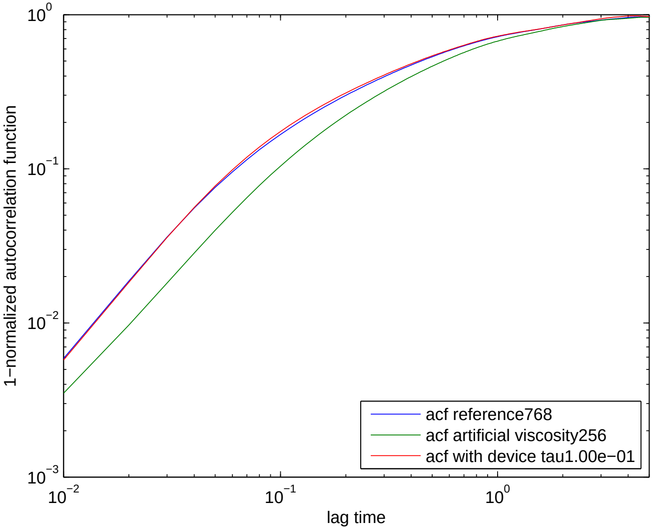
Auto-correlation functions

$$R_{\omega\omega}(\tau) = \frac{1}{T}\int_0^T \omega(t+\tau)\omega(t)\ \mathrm{d}t$$



Auto-correlation functions

$$1 - R_{\omega\omega}(\tau)$$



Summary

- Truncation of length scales disturbs spectrum at the smallest resolved scales
- A correction is made that restores the energy spectrum
- This correction also improves dispersivity and decorrelation times