Hermite Harmonics: A basis for spectral, global scale, ocean models (+ test cases)

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Layout

1. Theoretical background:

   Zonally propagating wave solutions obtained from an approximate Schrödinger equation of the Shallow Water Equations on a rotating sphere (focus: a "thin" ocean (gravity waves ~2-3 m/sec)

2. Numerical comparison: HH versus SH – Problematic baroclinic limit

3. Concluding remarks and test cases

Linearized Shallow Water Equations = Free LTE (inhomogeneous forcing of 3D dynamics determines *gH*)

$$\frac{\partial \underline{\mathbf{V}}}{\partial t} + f\hat{\mathbf{k}} \times \underline{\mathbf{V}} = -g\underline{\nabla}\eta$$

$$\frac{\partial \eta}{\partial t} = -H\underline{\nabla} \cdot \underline{\mathbf{V}}$$

<u>Scale:</u>

$\underline{\mathbf{V}}$: $2\Omega a$ (Earth's rotation frequency and radius, respectively)

η : Mean thickness H (so total height is $h=H+\eta$)

t : 2Ω

<u>Get</u>: A single nondimenasional parameter (the new g):

$\alpha = gH/(2\Omega a)^2$ (the inverse of Lamb #)

Assume zonally propagating wave solution: $e^{ik(\lambda - Ct)}$

$$\begin{bmatrix} 0 & \sin\phi & \dfrac{\alpha}{\cos\phi} \\ \dfrac{\sin\phi}{k^2} & 0 & \dfrac{\alpha}{k^2}\dfrac{\partial}{\partial\phi} \\ \dfrac{1}{\cos\phi} & \tan\phi - \dfrac{\partial}{\partial\phi} & 0 \end{bmatrix} \begin{bmatrix} u \\ V \\ \eta \end{bmatrix} = c \begin{bmatrix} u \\ V \\ \eta \end{bmatrix}$$

The above exact set can be transformed into a Schrödinger equation whose potential can be approximated (on earth $\alpha \ll 1$) which yields analytic solutions for the energy ($\Rightarrow$phase speed) and the meridional amplitude variation

Analytical expressions should be compared to exact (numerical) solutions of the exact system

$$\frac{\partial}{\partial\phi}\begin{bmatrix} V\cos\phi \\ \eta \end{bmatrix} = \begin{bmatrix} \dfrac{\sin\phi}{c\cos\phi} & \left(\dfrac{\alpha - c^2\cos^2\phi}{c\cos\phi}\right) \\ \dfrac{1}{\alpha}\left(\dfrac{k^2c^2 - \sin^2\phi}{c\cos\phi}\right) & -\dfrac{\sin\phi}{c\cos\phi} \end{bmatrix} \begin{bmatrix} V\cos\phi \\ \eta \end{bmatrix}$$

$$\begin{bmatrix} 0 & \sin\phi & \dfrac{\alpha}{\cos\phi} \\ \dfrac{\sin\phi}{k^2} & 0 & \dfrac{\alpha}{k^2}\dfrac{\partial}{\partial\phi} \\ \dfrac{1}{\cos\phi} & \tan\phi - \dfrac{\partial}{\partial\phi} & 0 \end{bmatrix} \begin{bmatrix} u \\ V \\ \eta \end{bmatrix} = c \begin{bmatrix} u \\ V \\ \eta \end{bmatrix}$$

$$V(\phi)\cdot\cos\phi = \psi(\phi) \times \left(\frac{\alpha - c^2\cos^2\phi}{c\cos\phi}\right)^{\frac{1}{2}}$$

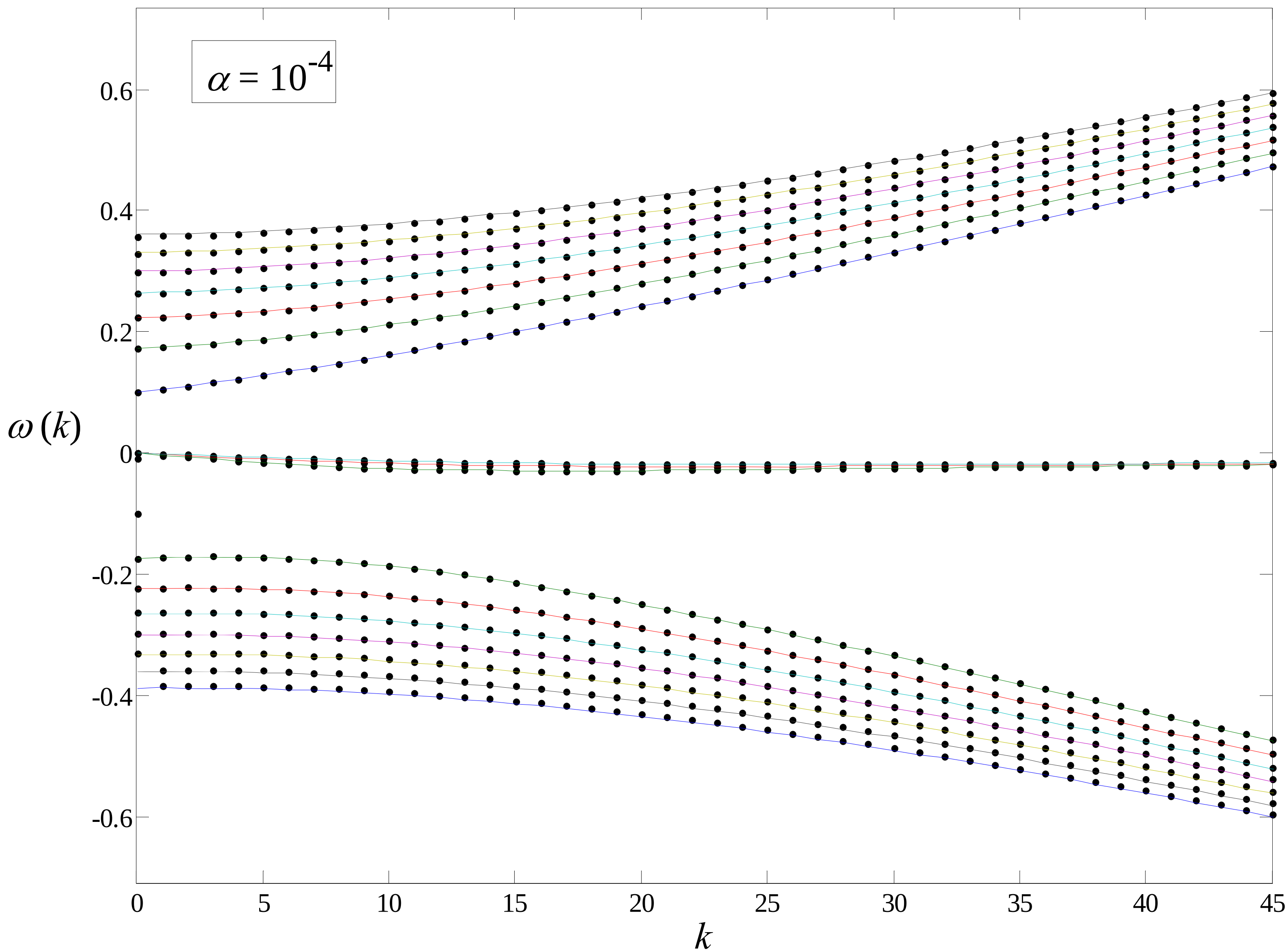
$$\frac{d^2\psi}{d\phi^2} - \left[\frac{1}{c} - \frac{k^2c^2}{\alpha} + \frac{k^2}{\cos^2\phi} + \frac{\sin^2\phi}{\alpha} + \frac{3}{4}\tan^2\phi\left(\frac{\alpha + c^2\cos^2\phi}{\alpha - c^2\cos^2\phi}\right)^2 - \frac{1}{2}\left(\frac{\alpha + c^2\cos^2\phi}{\alpha - c^2\cos^2\phi}\right) - \tan^2\phi\left(\frac{\alpha + 2c\cos^2\phi}{\alpha - c^2\cos^2\phi}\right)\right]\psi = 0$$

$$\alpha\frac{d^2\psi}{d\phi^2} + \left[E - \left(\sin^2\phi + \frac{\alpha k^2}{\cos^2\phi}\right)\right]\psi = 0$$

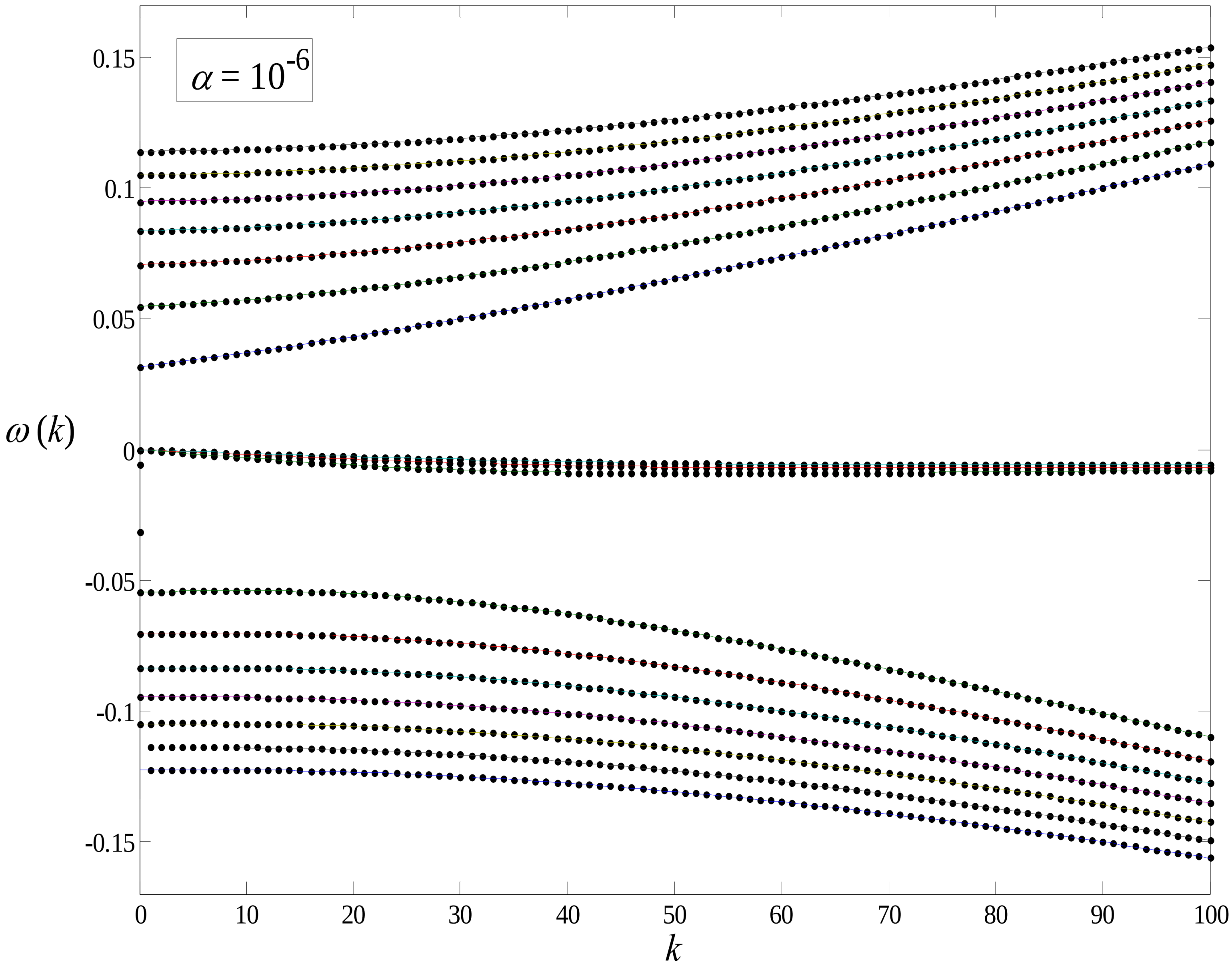
where $E = k^2c^2 - \dfrac{\alpha}{c}$

1. Boundary Conditions: regularity of V at the poles $\Rightarrow \psi(\phi = \pm\pi/2) = 0$
2. *E* in fourth problem (Lower-right box) yields 3 *c*'s via: $k^2c^3 - Ec - \alpha = 0$
3. At latitudes where $\alpha - c^2\cos^2\phi = 0$ the denominator of the spherical terms vanishes so $\alpha U_2(\phi)$ is not uniformly negligible

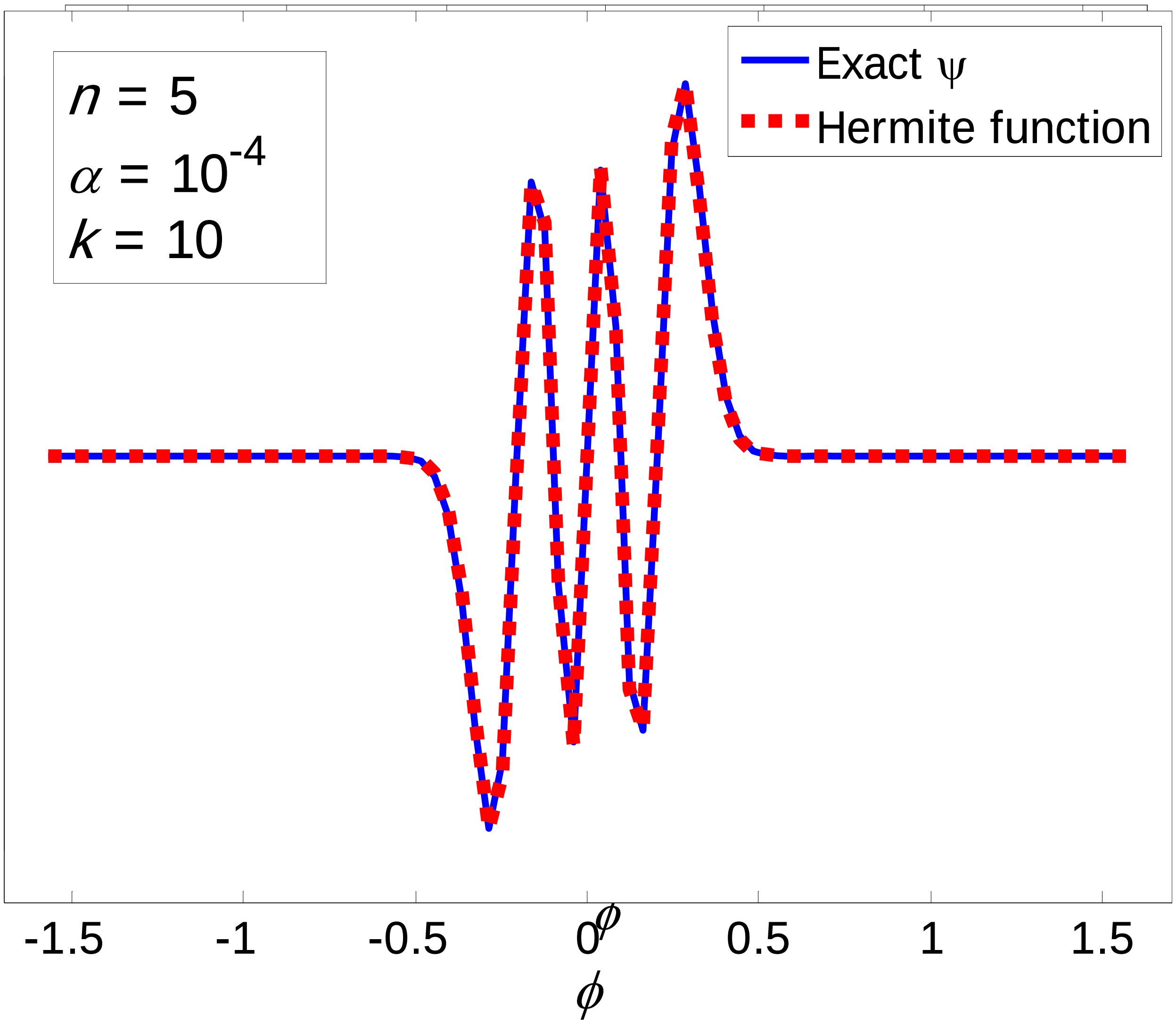
Comparison between the approximate and exact (numerical) dispersion relations ω(k)=kC(k)



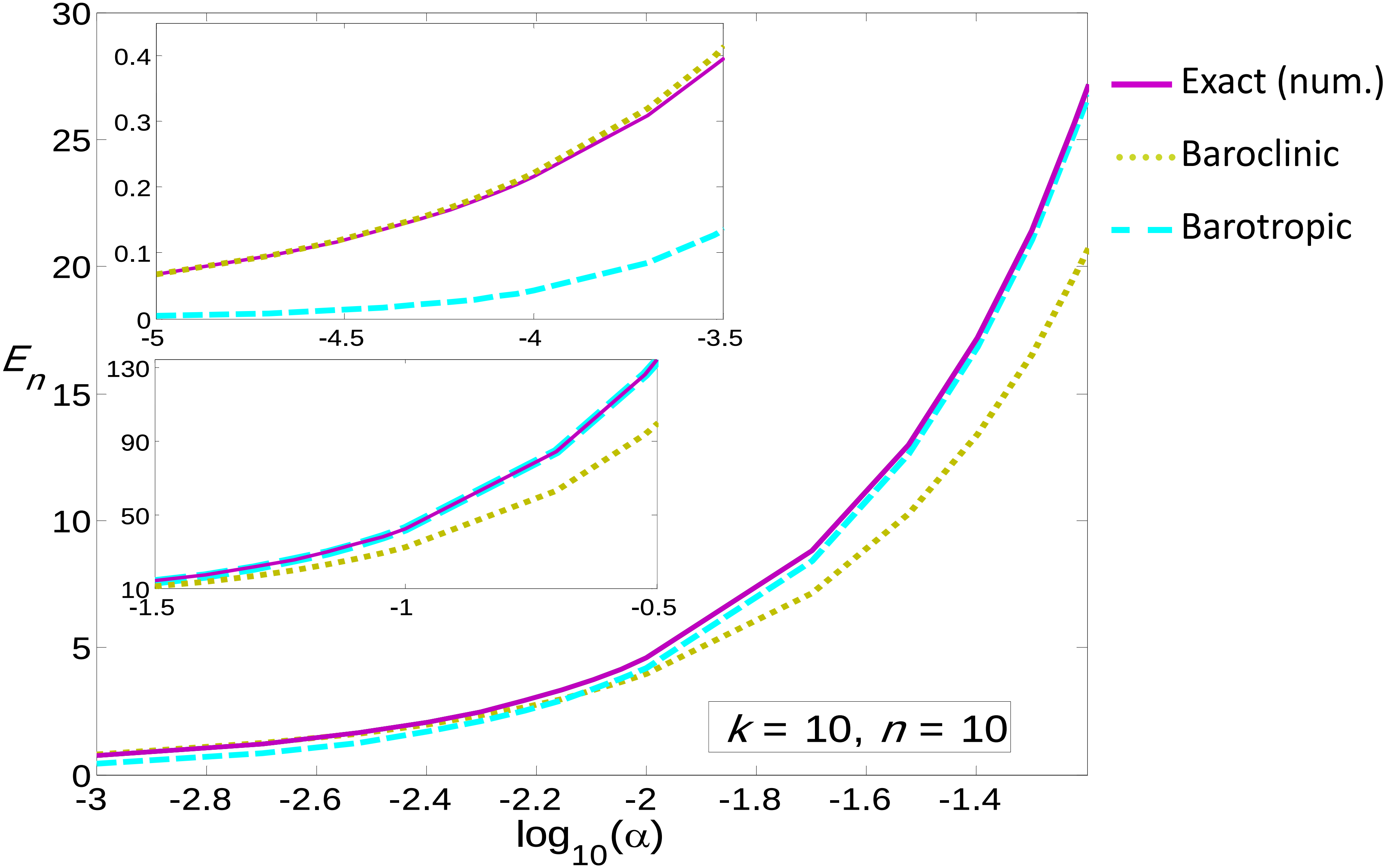
For smaller α (=10^{-6}) the approximation is accurate for higher n



The eigenfunction



The transition from baroclinic ($\alpha=5\cdot10^{-6}$) to barotropic ($\alpha=5\cdot10^{-2}$) theories:



The spectral method

Rewriting the LSWE in terms of the independent variable $\mu = \sin\phi$, and (the Robert functions) $V = v\cos\phi$ and $U = u\cos\phi$.

$$
\begin{aligned}
\frac{\partial U}{\partial t} &= \mu V - \alpha \frac{\partial \eta}{\partial \lambda} \\
\frac{\partial V}{\partial t} &= -\mu U - \alpha \left(1-\mu^2\right)\frac{\partial \eta}{\partial \mu} \\
\left(1-\mu^2\right)\frac{\partial}{\partial t}\eta &= -\frac{\partial U}{\partial \lambda} - \left(1-\mu^2\right)\frac{\partial V}{\partial \mu}
\end{aligned}
$$

The regularity of *u* and *v* at the poles implies that *U* and *V* must satisfy the boundary conditions $U(\mu=\pm 1) = V(\mu=\pm 1) = 0$, and these boundary conditions also ensure the regularity of the spatial derivatives of *U*, *V* and η.

The spectral method

Seeking approximate (truncated) series solutions of the form:

$$\xi(\lambda, \mu, t) = \sum_{m=-M}^{M} \sum_{n=0}^{N(m)} \xi_n^m(t)\,\varphi_n^m(\lambda, \mu)$$

where

- ξ stands for any of the dependent variables *U*, *V* or η
- ξ_n^m are the corresponding time dependent spectral coefficients
- φ_n^m are the spatial basis functions

The system of ODEs for the spectral coefficients

The system of ODEs for the spectral coefficients does not mix wave numbers, i.e. for a given wavenumber *m* the rate of change of the spectral coefficient ξ_n^m is independent of other wave numbers, so the system is solved one *m* at a time.

$$\mathbf{M^m}\frac{\mathbf{d}}{\mathbf{dt}}\mathbf{F^m} = \mathbf{C^m F^m}$$

where $\mathbf{F^m} = (U_0^m,\ldots,U_n^m, V_0^m,\ldots,V_n^m, \eta_0^m,\ldots, \eta_n^m)^T$ is the spectral coefficient vector for the given m, and $\mathbf{M^m}$ and $\mathbf{C^m}$ form the block matrices:

$$\underbrace{\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M9} \end{bmatrix}}_{\mathbf{M^m}} \begin{bmatrix} \dot{U}_0^m \\ \vdots \\ \dot{U}_N^m \\ \dot{V}_0^m \\ \vdots \\ \dot{V}_N^m \\ \dot{\eta}_0^m \\ \vdots \\ \dot{\eta}_N^m \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{C2} & \mathbf{C3} \\ \mathbf{C4} & \mathbf{0} & \mathbf{C6} \\ \mathbf{C7} & \mathbf{C8} & \mathbf{0} \end{bmatrix}}_{\mathbf{C^m}} \begin{bmatrix} U_0^m \\ \vdots \\ U_N^m \\ V_0^m \\ \vdots \\ V_N^m \\ \eta_0^m \\ \vdots \\ \eta_N^m \end{bmatrix}$$

The system of ODEs for the spectral coefficients

$$M9_{rs} = -\varepsilon_r^m \varepsilon_{r+1}^m \delta_{rs-2} + \left(1 - \varepsilon_{r-1}^m \varepsilon_{r-1}^m + \varepsilon_r^m \varepsilon_r^m\right)\delta_{rs} - \varepsilon_{r-1}^m \varepsilon_{r-2}^m \delta_{rs+2}$$

$$C2_{rs} = -C4_{rs} = \varepsilon_r^m \delta_{rs-1} + \varepsilon_{r-1}^m \delta_{rs+1}$$

$$C3_{rs} = \alpha C7_{rs} = -im\alpha\delta_{rs}$$

Hermite Harmonics:

$$\varepsilon_r^m = \sqrt{\frac{r}{2\Delta_m}} \qquad \Delta_m = \sqrt{\frac{1+\alpha m^2}{\alpha}}$$

$$C6_{rs} = \alpha C8_{rs} = \alpha\Delta\left[\varepsilon_r^m \varepsilon_{r+1}^m \varepsilon_{r+2}^m \delta_{rs-3} + \left(\frac{r-2}{2\Delta} - 1\right)\varepsilon_r^m \delta_{rs-1} - \left(\frac{r+1}{2\Delta} - 1\right)\varepsilon_{r-1}^m \delta_{rs+1} - \varepsilon_{r-1}^m \varepsilon_{r-2}^m \varepsilon_{r-3}^m \delta_{rs+3}\right]$$

Spherical Harmonics:

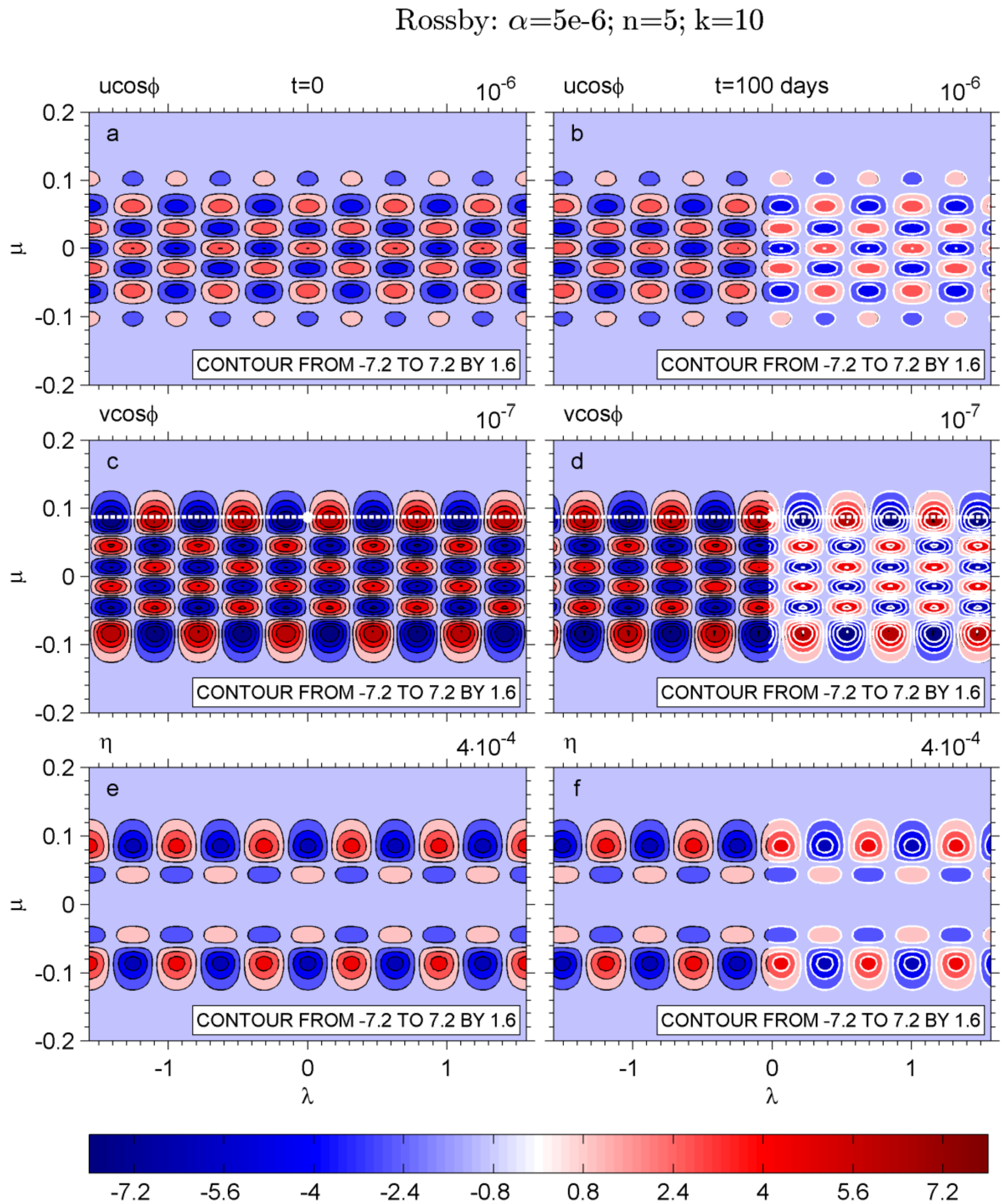
$$C6_{rs} = \alpha C8_{rs} = \alpha\left[-(1+r)\varepsilon_r^m \delta_{rs-1} + (r-2)\varepsilon_{r-1}^m \delta_{rs+1}\right]$$

$$\varepsilon_r^m = \sqrt{\frac{r^2 - m^2}{4r^2 - 1}}$$

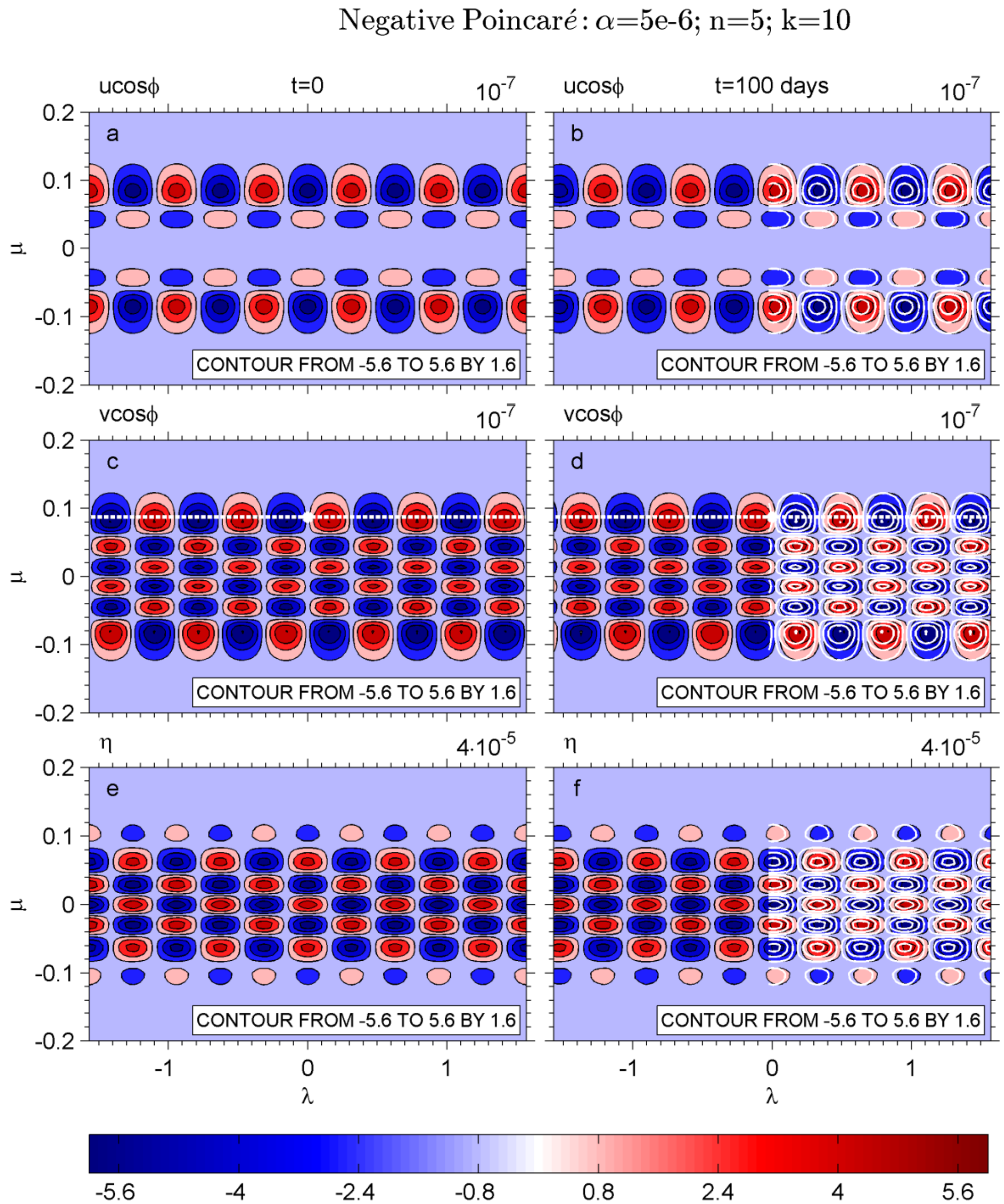
Single eigenmode test

The solver was initialized with a pure eigenmode of the baroclinic LSWE, i.e. with U,V, η corresponding to combination of a certain wavenumber, *k*, and a given mode number, *n*, where one of the three possible phase speeds determines the type of wave.

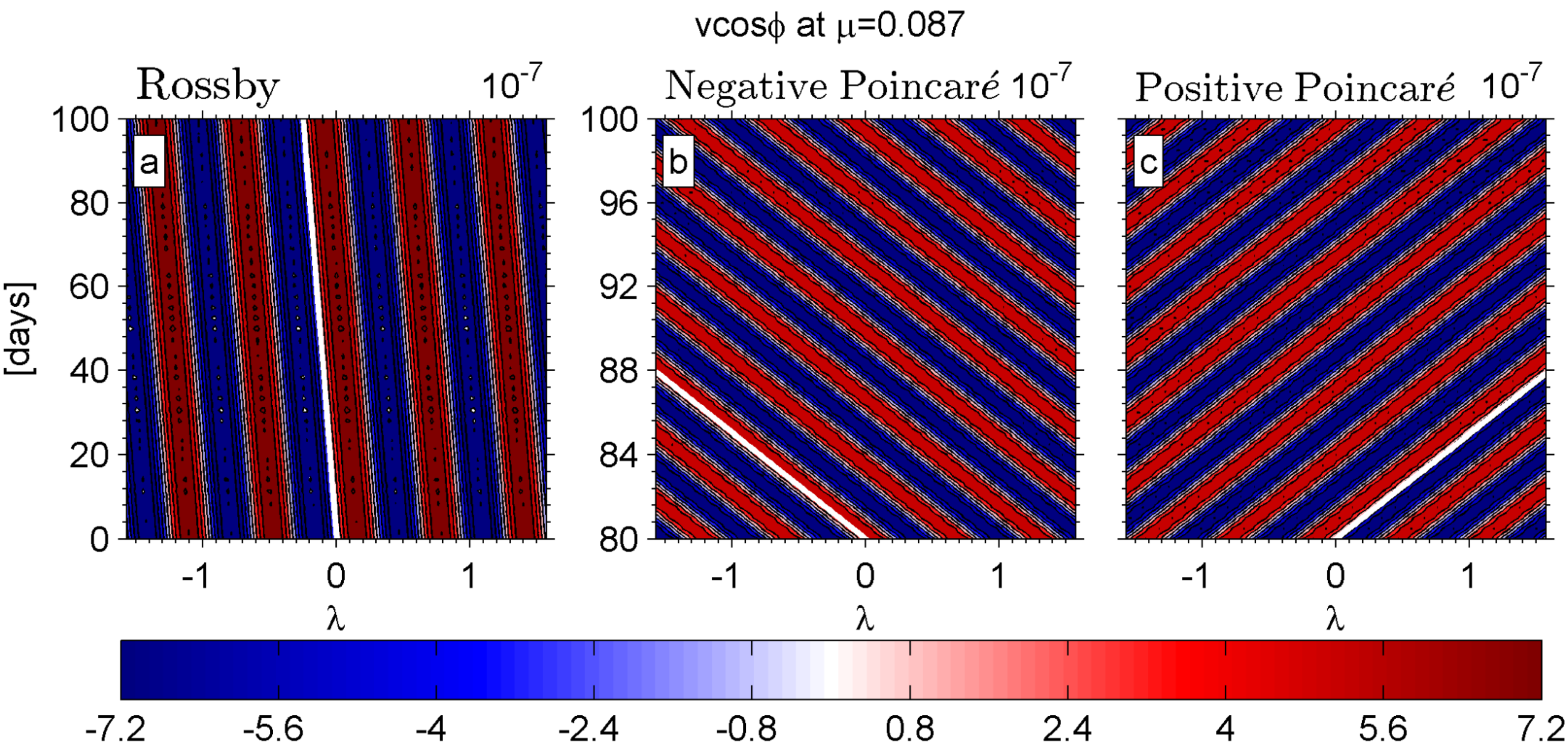
Single eigenmode test: HH



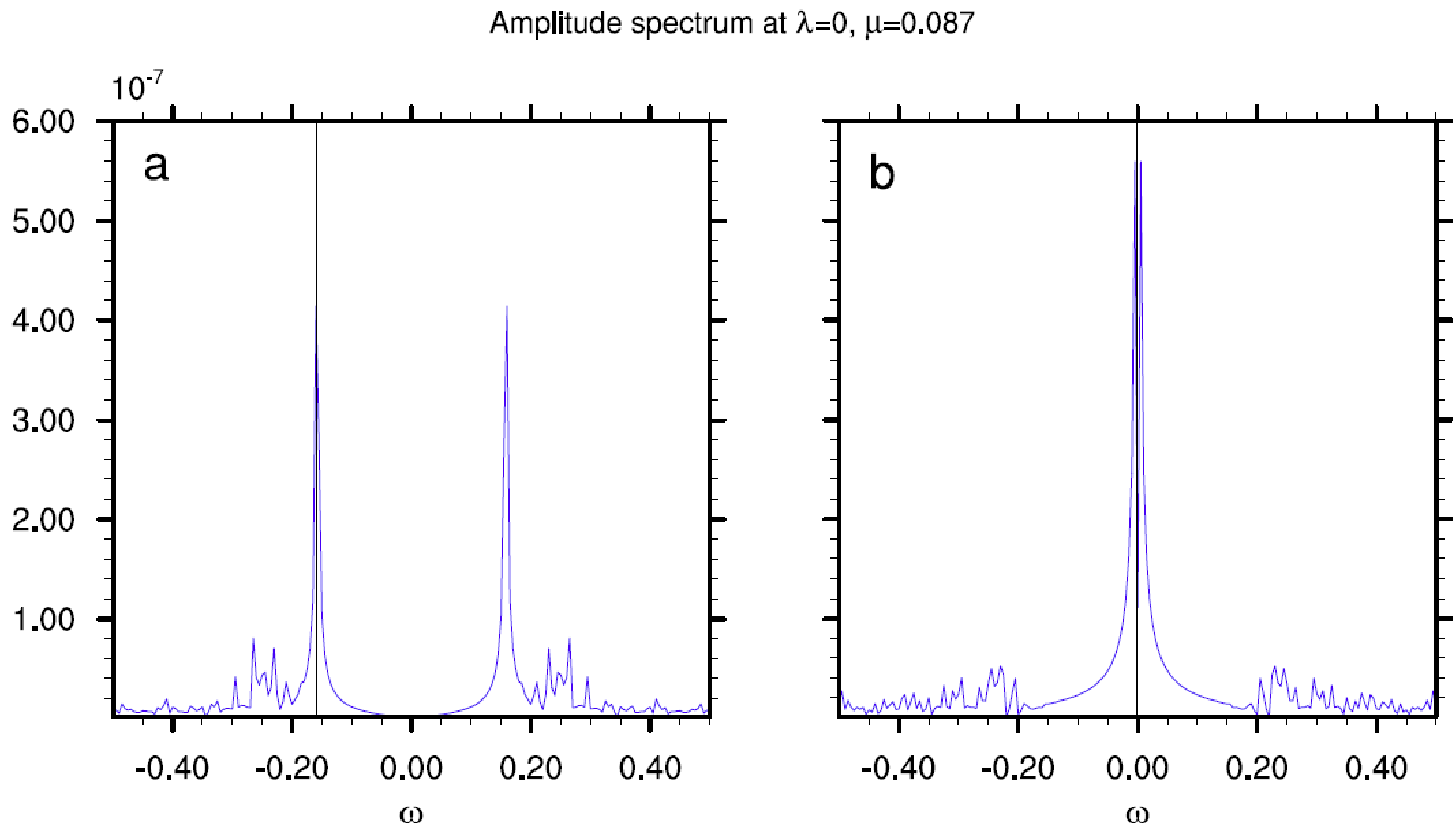
Single eigenmode test: HH



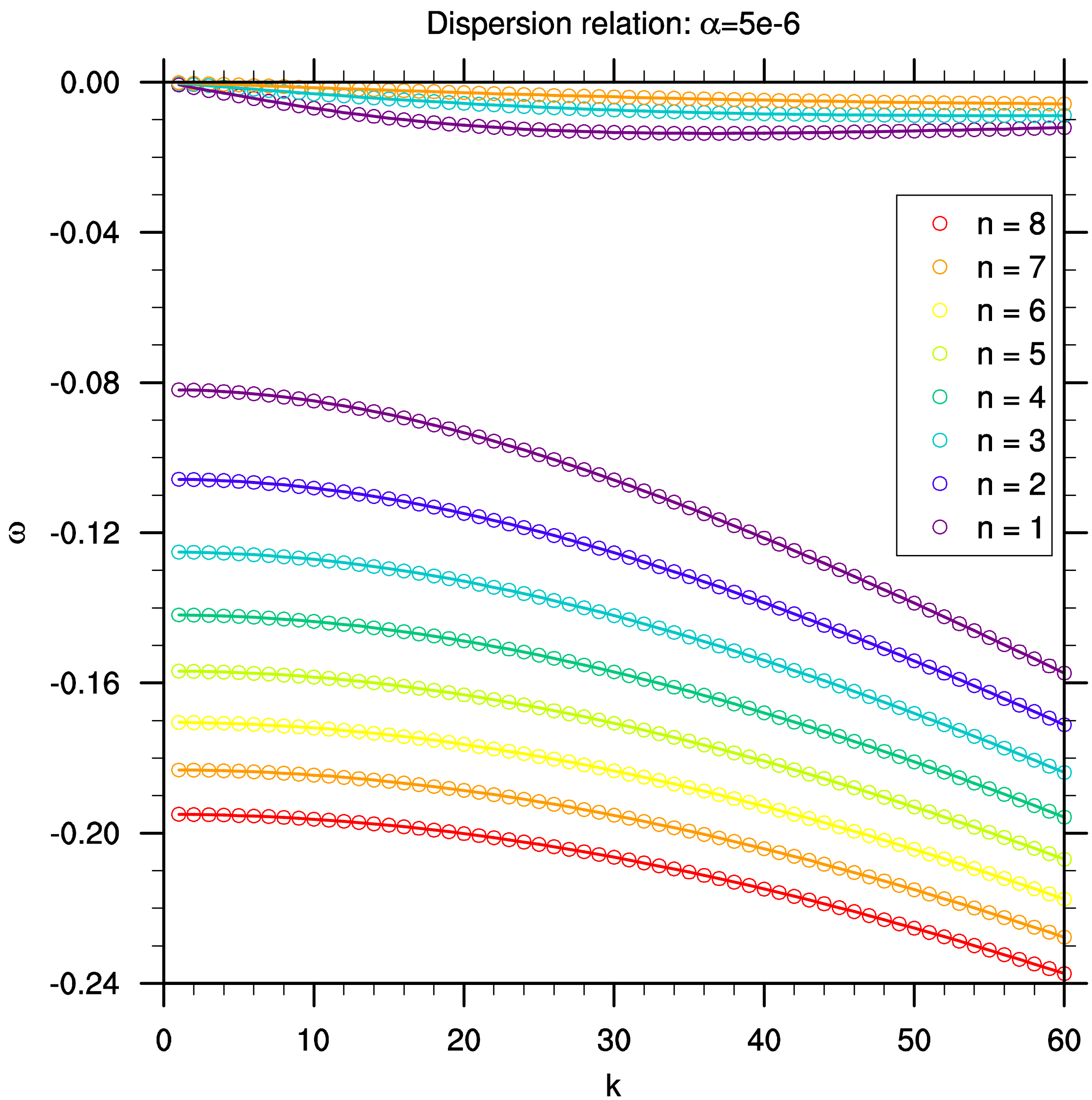
Single eigenmode test: HH – propagation



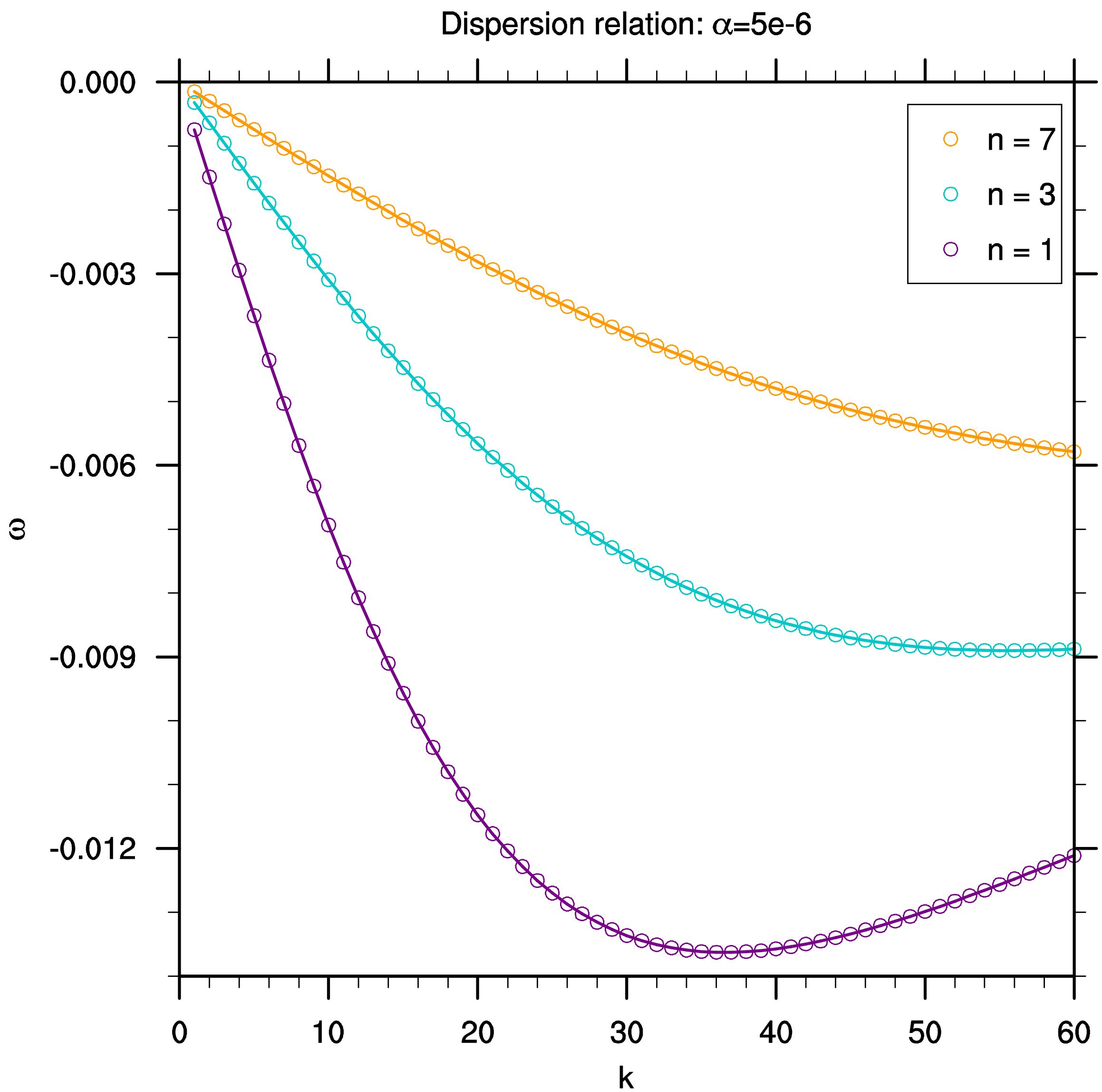
Single eigenmode test: HH



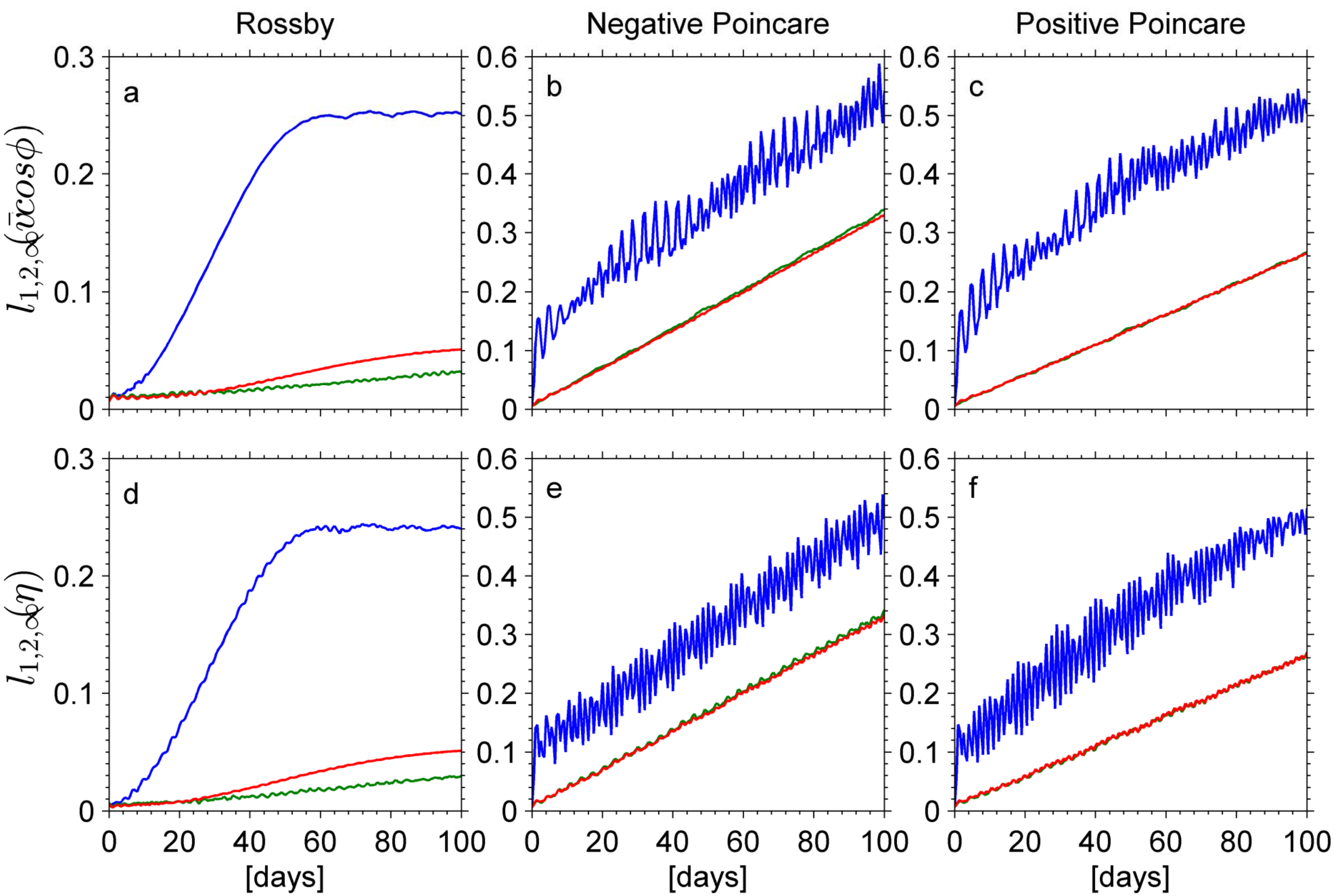
Single eigenmode test: HH



Single eigenmode test: HH



Single eigenmode test – l_1, l_2, l_∞ errors



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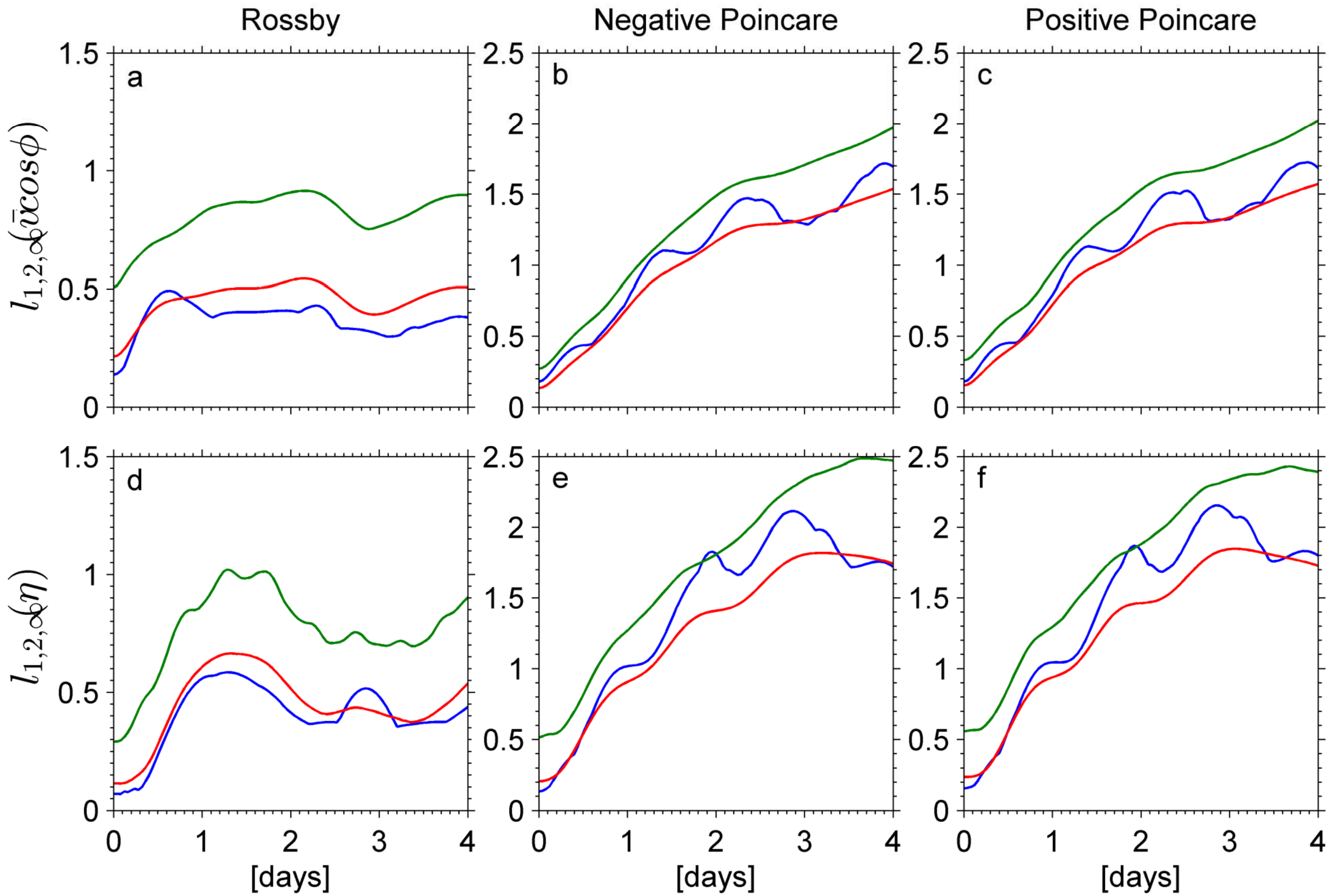

Comparison between HH and SH

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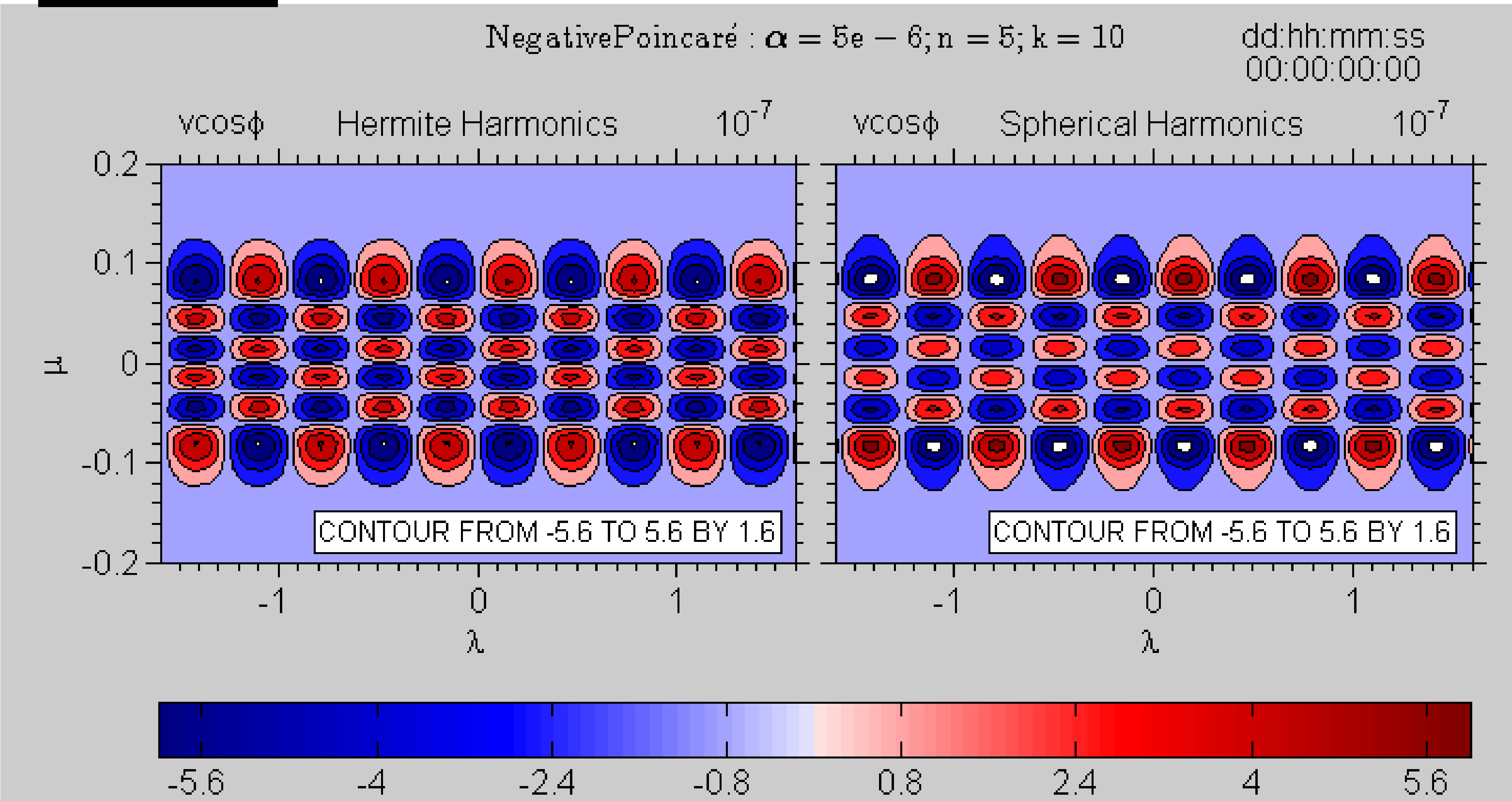

Comparison between HH and SH

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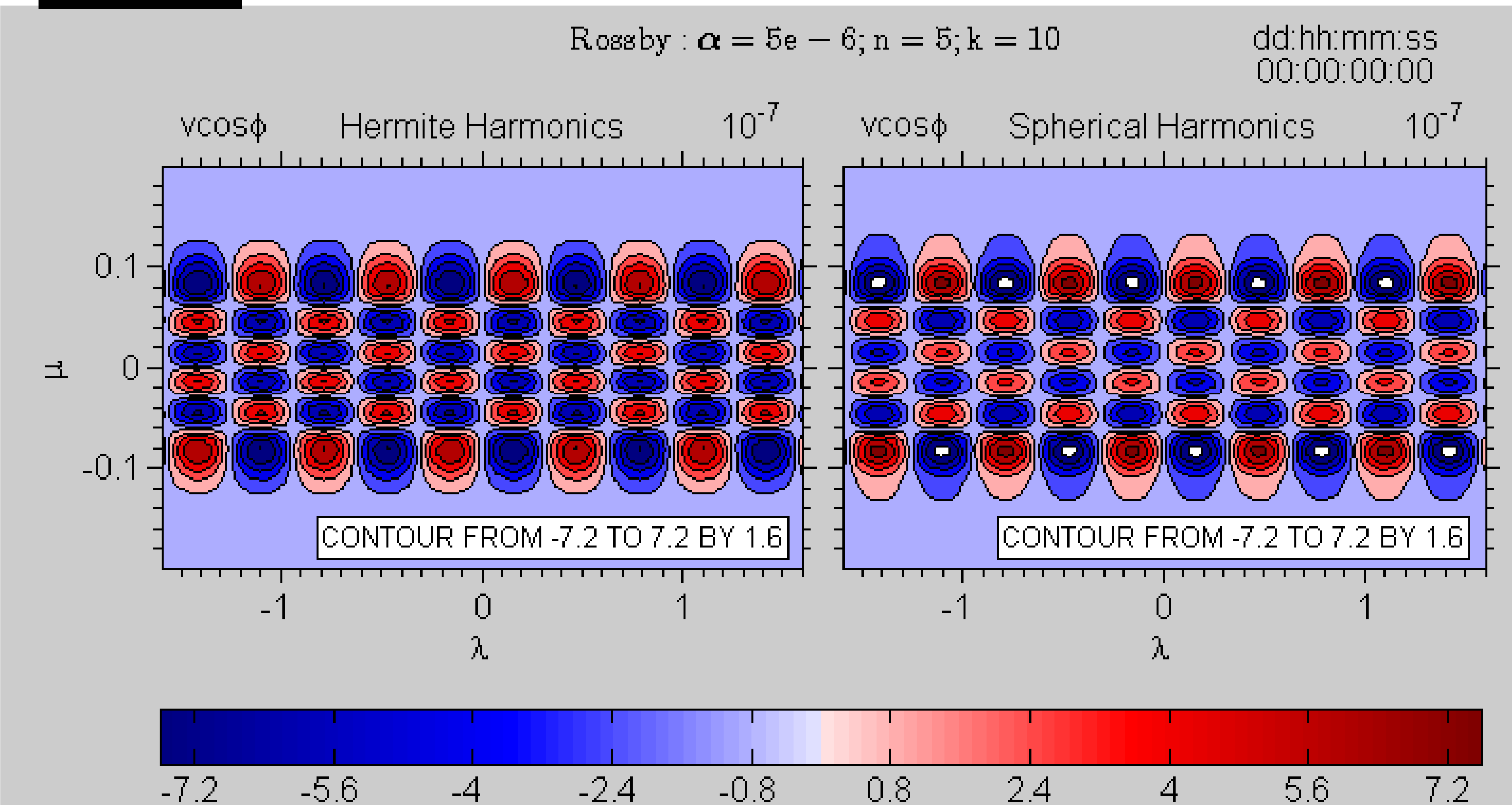

Summary

- The HSWS accurately conserve the eigensolutions of the LSWE to within 1% of the phase speed and with no appreciable distortion of the waves' structure for at least 100 day.
- In contrast, The LSWS exhibit severe distortion of the waves' structures within hours-days with comparable (or even slightly greater) spectral/grid-space resolutions and the solution looses its zonally propagating wave structure.
- This distortion, or tilting, phenomenon where the phase speed appears to be latitude dependent was previously noted (also with finite volume method - MITgcm) and was wrongfully attributed to physical phenomena.

Discussion

- The advantages of the proposed basis functions can be traced back to its origin in the system of ODEs for the spectral coefficients.

The system of ODEs for the spectral coefficients

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$$\mathbf{M^m}\frac{\mathbf{d}}{\mathbf{dt}}\mathbf{F^m} = \mathbf{C^m F^m}$$

where $\mathbf{F^m} = (U_0^m,\ldots,U_n^m, V_0^m,\ldots,V_n^m, \eta_0^m,\ldots, \eta_n^m)^T$ is the spectral coefficient vector for the given m, and $\mathbf{M^m}$ and $\mathbf{C^m}$ form the block matrices:

$$\underbrace{\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boxed{\mathbf{M9}} \end{bmatrix}}_{\mathbf{M^m}} \begin{bmatrix} \dot{U}_0^m \\ \vdots \\ \dot{U}_N^m \\ \dot{V}_0^m \\ \vdots \\ \dot{V}_N^m \\ \dot{\eta}_0^m \\ \vdots \\ \dot{\eta}_N^m \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{C2} & \mathbf{C3} \\ \mathbf{C4} & \mathbf{0} & \mathbf{C6} \\ \mathbf{C7} & \mathbf{C8} & \mathbf{0} \end{bmatrix}}_{\mathbf{C^m}} \begin{bmatrix} U_0^m \\ \vdots \\ U_N^m \\ V_0^m \\ \vdots \\ V_N^m \\ \eta_0^m \\ \vdots \\ \eta_N^m \end{bmatrix}$$

The system of ODEs for the spectral coefficients

Hermite Harmonics: for $\alpha n^2 \ll 1$ (and also $\alpha k^2 \ll 1$): $\Delta = \sqrt{\frac{1+\alpha k^2}{\alpha}} \to \frac{1}{\sqrt{\alpha}}$

$$M9_{rs} = -\underbrace{\frac{\sqrt{(r-2)(r-1)}}{2\Delta}}_{\sim n\sqrt{\alpha}}\delta_{sr-2} + \underbrace{\left(1-\frac{2r-1}{2\Delta}\right)}_{\sim 1-n\sqrt{\alpha}}\delta_{sr} - \underbrace{\frac{\sqrt{r(r+1)}}{2\Delta}}_{\sim n\sqrt{\alpha}}\delta_{sr+2}$$

M9 → I and consequently **M^m → I.**

Spherical Harmonics: for $\alpha n^2 \ll 1$ (and also $\alpha k^2 \ll 1$):

$$M9_{rs} = \varepsilon_{r-1}^m \varepsilon_{r-2}^m \delta_{sr-2} + \left(\varepsilon_r^m \varepsilon_r^m + \varepsilon_{r-1}^m \varepsilon_{r-1}^m - 1\right)\delta_{sr} + \varepsilon_r^m \varepsilon_{r+1}^m \delta_{sr+2}$$

ε_n^m is independent of α.

Discussion

- The advantages of the proposed basis functions can be traced back to its origin in the system of ODEs for the spectral coefficients.
- The Gaussian envelope of the Hermite functions makes the solver less sensitive to inaccuracies, such as round off or truncation errors, near the poles.
- **Hermite transforms** have high resolution (are denser) near the equator, in the waves' region of existence, whereas **Spherical Harmonics transforms** "waste" resolution near the poles, where the waves vanish.