Geometric cell alignment on geodesic grids

Pedro S. Peixoto
Saulo R. M. Barros

Applied Mathematics Department
University of São Paulo
Brazil

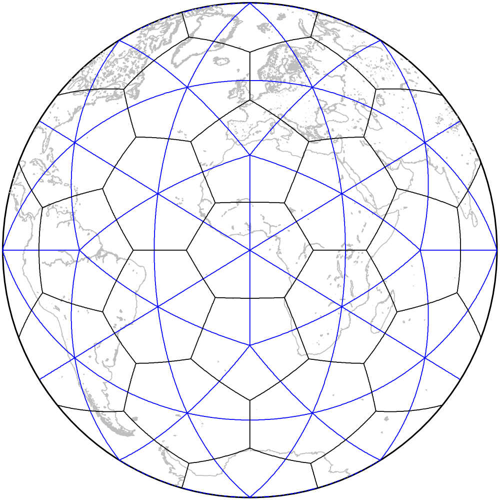
PDEs 2014 Talk

CAPES Funding is acknowledged

Summary

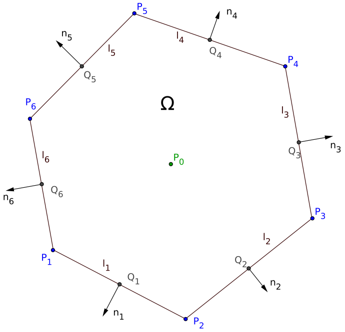
Icosahedral Grid

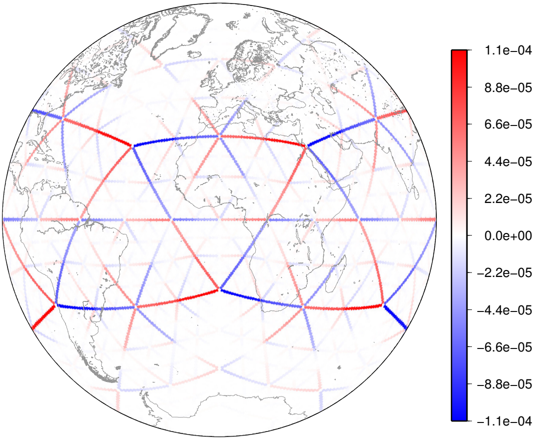
Icosahedral grid (Voronoi grid)



Finite-Volume Discretization

$$\operatorname{div}(\vec{v})(P_0) \approx \frac{1}{|\Omega|} \sum_{i=1}^{n} \vec{v}(Q_i) \cdot \vec{n}_i \, l_i.$$





Rotation vector field - Glevel 6 - 40962 nodes

Divergence Operator Discretization

Assuming the vector field given at the midpoints of the edges of the polygon, or precisely calculated:

$$\mathrm{div}(\vec{v})(P_0) \quad \approx \quad \frac{1}{|\Omega|}\int_\Omega \mathrm{div}(\vec{v})\, d\Omega = \frac{1}{|\Omega|}\int_{\partial\Omega} \vec{v}\cdot\vec{n}\, d\partial\Omega \approx \frac{1}{|\Omega|}\sum_{i=1}^{n} \vec{v}(Q_i)\cdot\vec{n}_i\, l_i.$$

On a plane:

- Average approximation: 2nd order if P_0 is the centroid
- Divergence approximation: 1st order only in general
- Rectangle: 2nd order
- Odd number of edges (triangle, pentagon): 1st order only, even if regular.
- General quadrilaterals and hexagons?

Geometric Alignment

Definition (Planar aligned polygon)

A polygon on a plane with an even number of vertices, given by $\{P_i\}_{i=1}^{n}$, is aligned if for each edge $e_i = \overline{P_i P_{i+1}}$ the corresponding opposite edge $e_{i+n/2} = \overline{P_{i+n/2} P_{i+n/2+1}}$ is parallel and has the same length as e_i.

Definition (Spherical aligned polygon)

A spherical polygon with an even number of edges is aligned if its radial projection onto the plane tangent to the sphere at its centroid is a planar aligned polygon.

OBS: Geodesics are straight lines in the projected plane

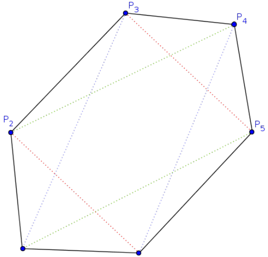
Alignment Index

Proposition (Alignment Index)

A polygonal cell Ω is aligned if, and only if, the nondimensional $\Xi(\Omega)$ is zero

$$\Xi(\Omega) = \frac{1}{n\bar{d}} \sum_{i=1}^{n/2} |d_{i+1+n/2,i} - d_{i+n/2,i+1}| + |d_{i+1,i} - d_{i+n/2+1,i+n/2}|$$

- $\bar{d} = \frac{1}{n} \sum_{i=1}^{n} d_{i,i+1}$
- $d_{i,j}$ is distance metric between polygon vertices P_i and P_j, $i, j = 1, ..., n$
- The greater Ξ, the greater miss-alignment



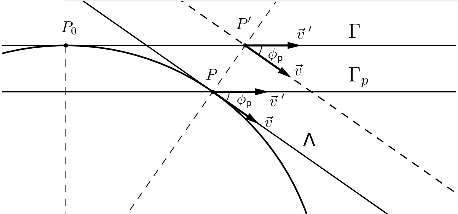
Main result on the sphere

Theorem

Let $\vec{v}$ be a $\mathcal{C}^4$ vector field on the sphere and Ω an aligned spherical polygon with n geodesic edges, diameter d and area $|\Omega|$ satisfying $\alpha d^2 \leq |\Omega| \leq d^2$, for some positive constant α. Then there is a constant C, which independs on the diameter d, such that

$$\left| div(\vec{v})(P_0) - \frac{1}{|\Omega|} \sum_{i=1}^{n} \vec{v}(Q_i) \cdot \vec{n}_i \, l_i \right| \leq Cd^2,$$

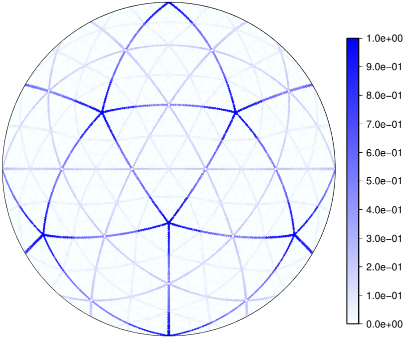
where P_0 is the mass centroid of Ω.



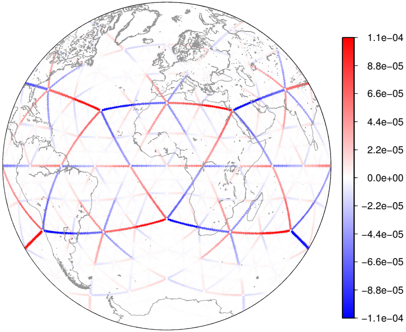
See Peixoto and Barros (2013)

Alignment Index

Alignment index



Div. of Rotation



Glevel 6 - 40962 nodes

Grid imprinting

For $\Xi < 1/100$:

glevel	% Align Cells
4	30.99%
5	49.22%
6	70.12%
7	84.24%
8	91.85%

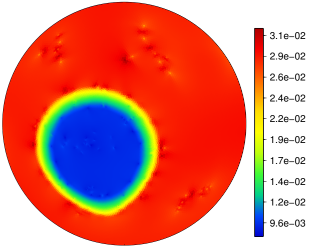
- Differences in cell geometry results in differences in the convergence orders of the discretization
- Aligned cells have faster convergence than non-aligned cells
- Badly aligned cells will have larger errors, and if these are related to the grid structure, we will have grid imprinting

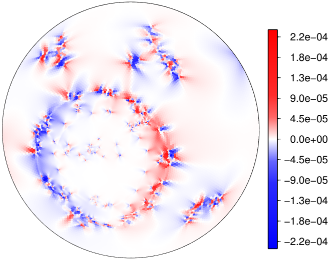
- Analogous theorems for Rotational (*curl*) and Laplacian operator
- Theory constructed generally for any geodesic grid

Locally refined SCVT grids

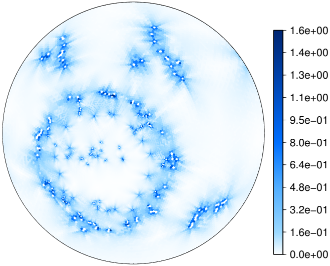
Grid - 40962 cells

Diameters:





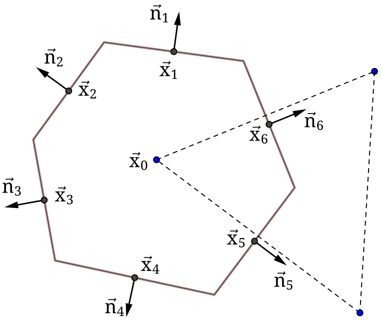
Div Rotation



Align. Ind.

Summary

Vector reconstruction



Analysed Methods:

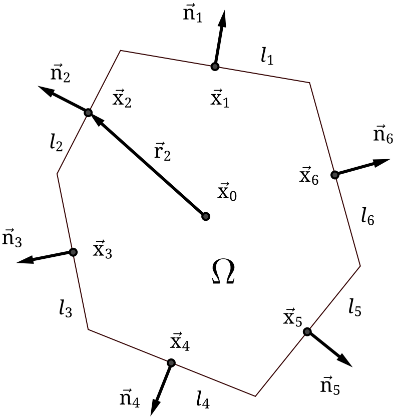
- Perot's method
- Klausen et al method (RT0 generalized to polygons)
- Polynomial (Least Sqrs.)
- RBF
- ⋆ Hybrid (Perot and Linear LSQ)

See Peixoto and Barros (2014) - under review - JCP Preprint

Perot's Method

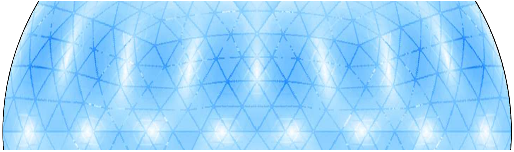
$$\vec{u}_0 = \frac{1}{|\Omega|} \sum_{i=1}^{n} \vec{r}_i \, u_i \, l_i,$$

- Divergence Theorem based
- Low cost
- Exact for constant fields
- 1st order only in general
- 2nd order on aligned cells

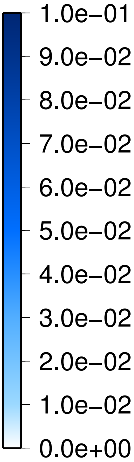


Hybrid scheme

PEROT - Max Error = 9.7E-2 RMS=1.3E-2



HYBRIB - Max Error = 4.3E-2 RMS=1.2E-2



Vector recon. to Voronoi cell nodes
Rossby-Haurwitz wave 8 - Icosahedral grid level 7
HYBRID: 84% Perot's method and 16% Linear LSQ method

Hybrid scheme

Perot's method on well aligned cells (majority) and linear LSQ method on ill aligned cells (minority)

- 2nd order accurate
- Low cost on fine grids (cost dominated by Perot's method)
- No pre-computing needed (less memory usage compared to RBF and LSQ)
- Applicable to any geodesic grid (tested in locally refined SCVT)

Application

2nd order semi-Lagrangian transport model for staggered Voronoi grids

Summary

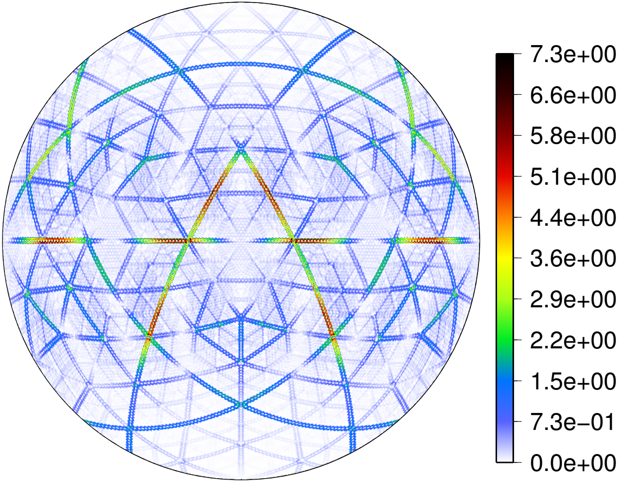
Conclusions

- Geometrical alignment:
  Better understanding of grid imprinting
- General Mathematical proofs:
  Plane and sphere for arbitrary polygons
- Alignment index:
  Tool for development of numerical methods
  Tool for grid development
- Analysis maybe be extended to other operators and dicretizations

Where are we going with this?

Analysis of shallow water model

Thuburn et al (2009) tangent vector reconstruction



Error of Rossby-Haurwitz wave 8 - Icos glevel 6

References

- Peixoto, PS and Barros, SRM. 2013. *Analysis of grid imprinting on geodesic spherical icosahedral grids*, J. Comput. Phys. 237 (March 2013), 61-78.
- Peixoto, PS and Barros, SRM. 2014. *On vector field reconstructions for semi-Lagrangian transport methods on geodesic staggered grids*, J. Comput. Phys. (under review)

Preprints available at www.ime.usp.br/∼pedrosp

Thank you very much!