Determining The Effective Resolution Of Advection Schemes

James Kent, Christiane Jablonowski, Jared Whitehead and Richard Rood

The effective resolution is the smallest scale that is completely resolved by the model

We develop two methods to calculate the effective resolution of advection schemes

- dispersion relation analysis
- numerical testing

We use these methods to investigate modeling choices: order-of-accuracy, explicit diffusion, time stepping, flux limiters

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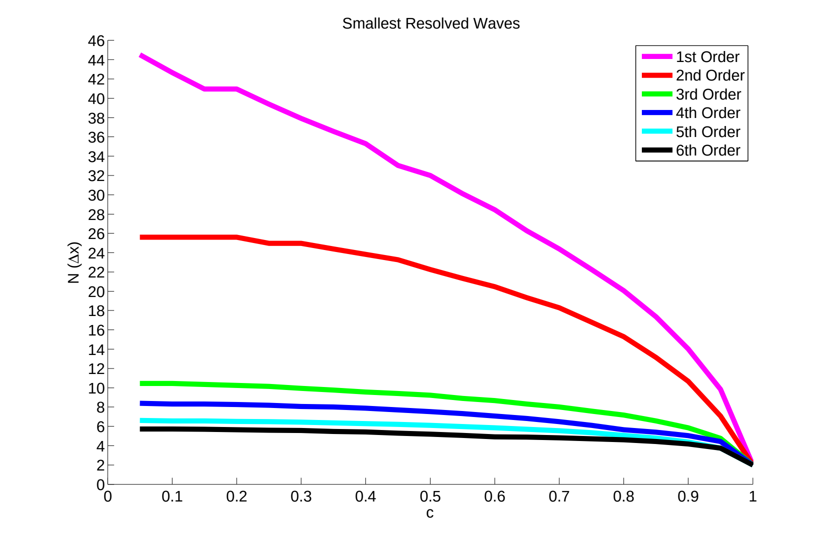

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Introduction

Advection schemes are important building blocks of atmospheric dynamical cores. The effective resolution of a numerical scheme is the smallest spatial scale, i.e. the largest wave number, that is fully resolved by that numerical scheme. The effective resolution is often significantly larger than the grid spacing, and it provides insight into which atmospheric features a model at a given grid resolution is able to represent.

One tool to evaluate numerical schemes is dispersion relation analysis. This analysis has been used to calculate the effective resolution of numerical schemes for the linearized shallow-water equations (Ullrich, 2014). We use this dispersion relation analysis to assess numerical advection schemes, taking into account the effects of different sized time steps.

The dispersion relation analysis can only be applied to linear schemes, yet many advection schemes make use of non-linear components, such as flux limiters. We therefore develop a numerical test method that enables the calculation of the effective resolution of non-linear schemes.

The advection equation is given as: $$\frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} = 0$$

Here q is the advected quantity (tracer mixing ratio) and u=1 is the constant velocity. We only consider a uniform grid of equal grid spacing with 0≤x≤1.

Dispersion Relation Analysis

The advection equation supports wavelike solutions of the form:

$$q = \hat{q}\exp(i(kx - \omega t))$$

Using this, the amplitude factor and dispersion relation for the advection equation can be calculated as:

$$|\Gamma| = 1 \qquad \omega = uk$$

To calculate the amplitude factor and dispersion relation of a given numerical scheme we insert the solution for the discrete tracer into the scheme's discretization:

$$q_j^n = \hat{q}\exp(i(kx_j - \omega t_n))$$

Wave number k is classed as resolved by a numerical scheme if the amplitude factor and dispersion relation are within 99% of the true value. We perform the analysis for a number of Courant numbers, c = uΔt/Δx. For consistency we evaluate the cumulative effect of the schemes over the distance Δx. This means that the numerically calculated amplitude factor is taken to the power 1/c. This process is described fully in Kent et al. 2014a.

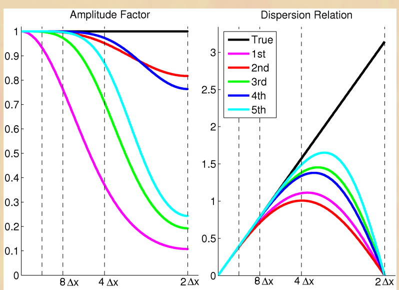

Figure 1: The amplitude factor and dispersion relation for the first- to sixth-order forward-in-time finite-difference schemes, plotted against wave length for c=0.1. The vertical dotted lines indicate the 16Δx, 8Δx, 4Δx and 2Δx wave lengths. The effective resolution of a numerical scheme is the largest wave number where both the amplitude factor and dispersion relation are within 99% of the true value.

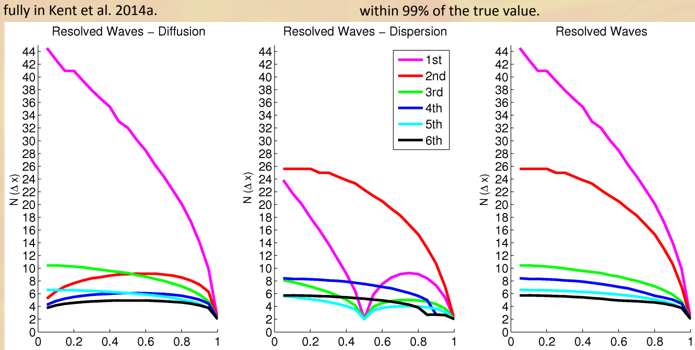

Figure 2: The smallest resolved wave in terms of N (Δx) due to the diffusion/amplitude factor error (left), the dispersion relation error (center), and both the diffusion and dispersion error (right), for the first- to sixth-order forward-in-time finite-difference schemes for 0<c≤1.

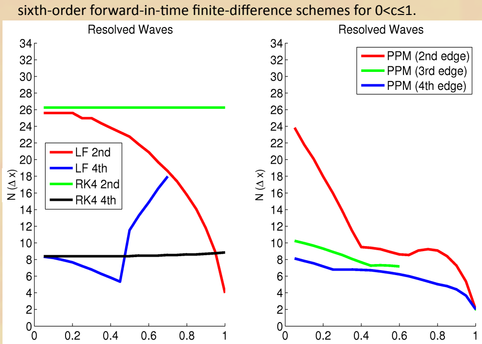

Figure 3: The effective resolution, in terms of N (Δx), of leapfrog (LF) and fourth-order Runge-Kutta (RK4) timestepping with second- and fourth-order spatial discretizations (left) and of the Piecewise Parabolic Method (PPM) without limiting, with second-, third- and fourth-order edge reconstructions. The lines end if the scheme becomes unstable.

Numerical Test

We have developed a test that allows the calculation of the effective resolution of any advection scheme (linear or non-linear). To create initial conditions of wave number k we use: $$q_0 = 1 + \cos(2\pi kx)$$

Using a given numerical scheme we advect this profile over a number of time steps. We calculate the normalized mean square error and decompose this into diffusive and dispersive parts following Takacs (1985). If the diffusive or dispersive error exceeds a given threshold, then wave number k is classed as unresolved. The process is repeated over all wave numbers k until the first unresolved wave is found. The numerical test is described in more detail in Kent et al. 2014b.

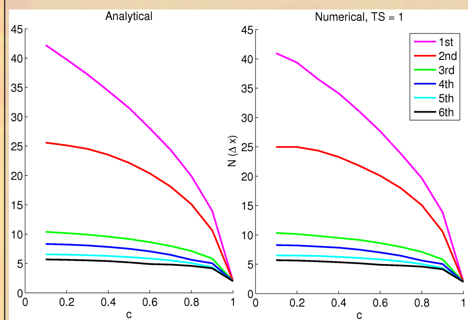

Figure 4: Comparing the analytical dispersion analysis method with the numerical test method. The left plot shows the effective resolution of the first- to sixth-order forward-in-time finite-difference schemes calculated using the dispersion analysis method (see Figure 2). The right plot shows the effective resolution of the same schemes but calculated using the numerical testing over one time step (TS). This confirms the accuracy of the numerical test method to calculate effective resolution.

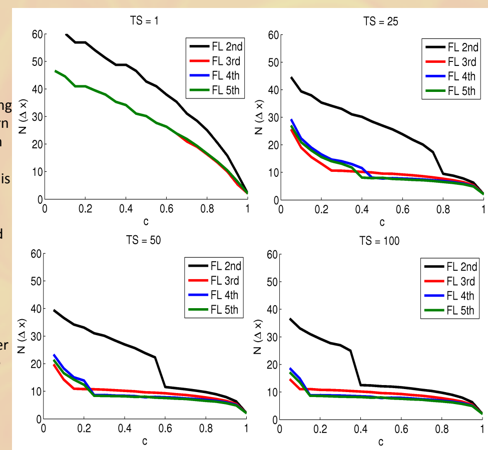

Figure 5: The effective resolution of the second- to fifth-order forward-in-time finite-difference schemes using the flux limiter (FL) of Thuburn (1996). The results are shown for a number of different length simulations, where TS is the number of time steps. Initially the flux limiter introduces large diffusion and dispersion errors, and hence has a poor initial effective resolution. Over more time steps the effective resolution of the limited schemes improves, although they never improve beyond the effective resolution of the corresponding unlimited scheme.

Conclusions

The effective resolution is the smallest wave that a numerical scheme can fully resolve. This can be calculated for linear schemes using dispersion relation analysis. We have developed a test that enables the effective resolution of any scheme (linear or non-linear) to be calculated.

- Increasing the spatial order-of-accuracy improves the effective resolution
- The greatest improvement when increasing the order-of-accuracy is found for low order schemes. The improvement diminishes for higher-order schemes (above third-order)
- For non-linear schemes the effective resolution is dependent on the length of the simulation
- Initially, flux limiters introduce large errors, and the effective resolution is poor. Over more time steps the effective resolution of the limited schemes tends to that of the corresponding order unlimited scheme

The effective resolution of numerical advection schemes is always larger than the grid spacing. For a third-order scheme at low Courant numbers the effective resolution is close to 10 times the grid spacing. Although the advection scheme is far removed from the dynamical core, the gap between the grid-scale and the effective resolution provides an insight into the description of uncertainty that is associated with dynamical cores (e.g. variable resolution grids, topography, grid-scale physics).

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