Generalization of Arakawa's Jacobian

Chiara Sorgentone[1]
Irene Milillo[1], Sandro Calmanti[2], Bernardo Favini[1]

[1] Università 'Sapienza' di Roma; [2]ENEA UTMEA-CLIM

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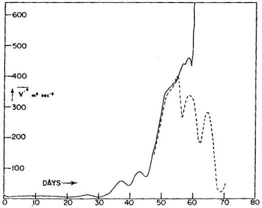
Introduction

- Phillips (1956) observed a non-linear computational instability
- The instability was unaffected by a reduction in the time step
- The magnitude of the solution values did not simply grow exponentially but instead remained relatively small and nearly constant before exploding into astronomical values
- ALIASING: non-linear problems would need an infinite spectrum but in discrete problems the spectrum is finite $\rightarrow$ the grid system incorrectly interprets wave lengths shorter than about 2 grid intervals as long waves
- In order to absorb the falsely generated energy, Phillips proposed the use of a smoothing process



Arakawa's idea

- In order to overcome the nonlinear numerical instability:
  - Use of a higher order method to reduce errors due to subscale processes
  - Use of energy-enstrophy conserving numerical methods (Arakawa, 1966)
- Discrete system which is physically analogous to the continuous system
- $\frac{dK}{dt} = \frac{d}{dt} \sum_n K_n, \quad K_n = \frac{1}{2}\overline{(\nabla \psi_n)^2}$
  (conservation on mean kinetic energy)
- $\frac{dV}{dt} = \frac{d}{dt} \sum_n V_n, \quad K_n = \frac{1}{2}\overline{(\nabla^2 \psi_n)^2} = k_n^2 K_n$
  (conservation on mean square vorticity)

  $\rightarrow$ Conservation of the average wave number, k, defined by $k^2 = \frac{\sum_n k_n^2 K_n}{\sum_n K_n}$

  $\rightarrow$ no systematic one-way cascade of energy into shorter waves can occur in two-dimensional incompressible flow

The vorticity equation

We consider the vorticity equation for two-dimensional incompressible flow:

$$\begin{cases} \nabla \cdot v = 0 \\ \zeta_t + v \cdot \nabla\zeta = 0 \\ \zeta = k \cdot \nabla \times v \end{cases}$$

The continuity equation implies:

$$v = k \times \nabla\psi \quad \rightarrow \quad \zeta = \nabla^2\psi$$

We introduce the Jacobian operator $J(a,b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}$, and we read again: $\frac{\partial \zeta}{\partial t} = J(\zeta, \psi)$.

Conserved quantities*:

- mean kinetic energy $E = \frac{1}{2}\overline{v^2} = \frac{1}{2}\overline{(\nabla\psi)^2} = \frac{1}{2}\overline{(\nabla\psi)\cdot(\nabla\psi)} = -\frac{\overline{\psi\zeta}}{2}$,
- mean square vorticity $G = \frac{\overline{\zeta^2}}{2}$

To see the conservation of energy we recall the laplacian property $\overline{f(\nabla^2 g)} = \overline{g(\nabla^2 f)}$; we get:

$$\overline{(\frac{\partial E}{\partial t})} = \frac{(\overline{-\psi\zeta_t - \zeta\psi_t})}{2} = \frac{[\overline{\psi J(\psi,\zeta)} + \overline{\zeta\nabla^{-2}J(\psi,\zeta)}]}{2} = \frac{[\overline{\psi J(\psi,\zeta)} + \overline{\nabla^{-2}\zeta J(\psi,\zeta)}]}{2} = \overline{\psi J(\psi,\zeta)} = 0$$

And for the conservation of enstrophy:

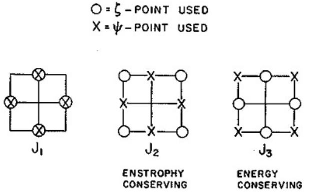
$$\overline{(\frac{\partial G}{\partial t})} = \overline{(\frac{\partial}{\partial t}(\frac{\zeta^2}{2}))} = \overline{(\zeta\frac{\partial \zeta}{\partial t})} = \overline{(-\zeta J(\psi,\zeta))} = 0$$

$*\overline{f} = \frac{1}{|\Omega|}\int_\Omega f dxdy$, Ω biperiodic domain

Arakawa's Jacobian

$$\mathbf{v} = (u, v),\ \ u = -\frac{\partial \psi}{\partial y},\ \ v = \frac{\partial \psi}{\partial x}; \qquad \zeta = \nabla \times \mathbf{v} = \nabla^2 \psi$$

1. $J_1(\zeta, \psi) = \zeta_x \psi_y - \zeta_y \psi_x$
2. $J_2(\zeta, \psi) = -\frac{\partial}{\partial_x}(\zeta_y \psi) + \frac{\partial}{\partial y}(\zeta_x \psi)$
3. $J_3(\zeta, \psi) = \frac{\partial}{\partial_x}(\zeta \psi_y) - \frac{\partial}{\partial y}(\zeta \psi_x)$



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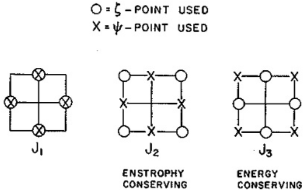

1. $(J_1)_{i,j} = \frac{1}{4d^2}[(\zeta_{i+1,j} - \zeta_{i-1,j})(\psi_{i,j+1} - \psi_{i,j-1}) - (\zeta_{i,j+1} - \zeta_{i,j-1})(\psi_{i+1,j} - \psi_{i-1,j})]$
2. $(J_2)_{i,j} = \frac{1}{4d^2}[-(\zeta_{i+1,j+1} - \zeta_{i+1,j-1})\psi_{i+1,j} + (\zeta_{i-1,j+1} - \zeta_{i-1,j-1})\psi_{i-1,j} + (\zeta_{i+1,j+1} - \zeta_{i-1,j+1})\psi_{i,j+1} - (\zeta_{i+1,j-1} - \zeta_{i-1,j-1})\psi_{i,j-1}]$
3. $(J_3)_{i,j} = \frac{1}{4d^2}[(\psi_{i+1,j+1} - \psi_{i+1,j-1})\zeta_{i+1,j} - (\psi_{i-1,j+1} - \psi_{i-1,j-1})\zeta_{i-1,j} - (\psi_{i+1,j+1} - \psi_{i-1,j+1})\zeta_{i,j+1} + (\psi_{i+1,j-1} - \psi_{i-1,j-1})\zeta_{i,j-1}]$

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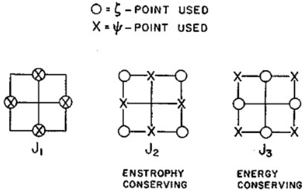

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$$J_{Arakawa} = \frac{1}{3}(J_1 + J_2 + J_3)$$

The main idea

- **Algebraic Construction**

  We consider a general finite differences discretization of a general operator which uses a 9x9 grid

$$
\begin{aligned}
J(\psi,\zeta) = {} & (A_1\psi_{i+1,j+1} + A_2\psi_{i+1,j} + A_3\psi_{i+1,j-1} + A_4\psi_{i,j+1} + A_5\psi_{i,j} + A_6\psi_{i,j-1} \\
& + A_7\psi_{i-1,j+1} + A_8\psi_{i-1,j} + A_9\psi_{i-1,j-1}) \cdot (A'_1\zeta_{i+1,j+1} + A'_2\zeta_{i+1,j} + A'_3\zeta_{i+1,j-1} \\
& + A'_4\zeta_{i,j+1} + A'_5\zeta_{i,j} + A'_6\zeta_{i,j-1} + A'_7\zeta_{i-1,j+1} + A'_8\zeta_{i-1,j} + A'_9\zeta_{i-1,j-1}).
\end{aligned}
$$

Requirements:

(a) Skew-symmetric property

(b) Conservation of energy

(c) Conservation of enstrophy

(d) Consistency (order two)

The linear system

We can see this non linear problem as a linear problem with 81 unknowns: $B_1 = A_1 A_1'$, $B_2 = A_1 A_2'$, .., then we read the general Jacobian as:

$$\sigma_i J_i(\zeta, \psi) = \sum_{i'} \sum_{i''} c_{i,i',i''} \zeta_{i+i'} \psi_{i+i''} \tag{1}$$

It is useful to read again the Jacobian as:

$$\sigma_i J_i(\zeta, \psi) = \sum_{i'} a_{i,i+i'} \zeta_{i+i'}, \quad where \quad a_{i,i+i'} = \sum_{i''} c_{i,i',i''} \psi_{i+i''} \tag{2}$$

or

$$\sigma_i J_i(\zeta, \psi) = \sum_{i''} b_{i,i+i''} \psi_{i+i''} \quad where \quad b_{i,i+i''} = \sum_{i'} c_{i,i',i''} \zeta_{i+i'} \tag{3}$$

in order to translate the properties of the analytic Jacobian in terms of properties of the numerical scheme for the 81 coefficients $\rightarrow$

The equations

(a) to obtain the skew-symmetric property, $J(\zeta,\psi) = -J(\psi,\zeta)$:

$$\sum_{i'}\sum_{i''} c_{i,i',i''}\zeta_{i+i'}\psi_{i+i''} = -\sum_{i'}\sum_{i''} c_{i,i',i''}\zeta_{i+i''}\psi_{i+i'} \Rightarrow c_{i,i',i''} = -c_{i,i'',i'}$$

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(b) to obtain conservation of enstrophy, we need $\overline{\zeta J(\psi, \zeta)} = 0$:

$$\sum_{i} \Delta x^2 \zeta_i J_i(\zeta, \psi) = \sum_{i}\sum_{i'} a_{i,i+i'}\zeta_i\zeta_{i+i'} = 0 \Rightarrow a_{i+i',i} = -a_{i,i+i'}$$

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(c) to obtain conservation of energy, we need $\overline{\psi J(\psi,\zeta)} = 0$:

$$\sum_i \Delta x^2 \psi_i J_i(\zeta,\psi) = \sum_i\sum_{i''} b_{i,i+i''}\psi_i\psi_{i+i''} = 0 \Rightarrow b_{i+i',i} = -b_{i,i+i'}$$

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(d) to obtain consistency, using Taylor expansions with the help of a symbolic manipulator, we nullify any equation proportional to dx^0, dx^1, dx^3, and we save only the contribution proportional to dx^2 of the Jacobian, meaning $(\frac{\partial\zeta}{\partial x}\frac{\partial\psi}{\partial y})$ and $(\frac{\partial\zeta}{\partial y}\frac{\partial\psi}{\partial x})$.

The General Scheme

By imposing the conditions a. b. c. d. we obtain a matrix of rank 80, corresponding to ∞^1 solutions. We proved the following theorem:

Theorem: Non-uniqueness of Arakawa's Jacobian

There exists a whole set of solutions for the numerical Jacobian which satisfy conservation of energy, conservation of enstrophy and skew-symmetric property; this set depends on one parameter (when the parameter is zero, we recover Arakawa's solution).

$$\boxed{\mathrm{J}_b(\zeta, \psi) = \vec{\psi}^T B \vec{\zeta}}$$

where

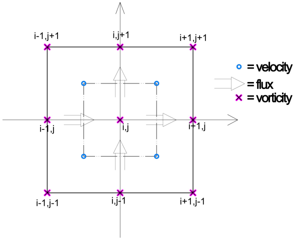
$$B = \begin{pmatrix} 0 & -(b-1) & 0 & (b-1) & 0 & 0 & 0 & 0 & 0 \\ (b-1) & 0 & (b+1) & -(b+1) & 0 & -(b-1) & 0 & 0 & 0 \\ 0 & -(b+1) & 0 & 0 & 0 & b+1 & 0 & 0 & 0 \\ -(b-1) & (b+1) & 0 & 0 & 0 & 0 & -(b+1) & (b-1) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (b-1) & -(b+1) & 0 & 0 & 0 & 0 & (b+1) & -(b-1) \\ 0 & 0 & 0 & b+1 & 0 & 0 & 0 & -(b+1) & 0 \\ 0 & 0 & 0 & -(b-1) & 0 & -(b+1) & b+1 & 0 & (b-1) \\ 0 & 0 & 0 & 0 & 0 & (b-1) & 0 & -(b-1) & 0 \end{pmatrix}$$

Conservative form of the general scheme

We look now for the conservative formulation of the scheme:

$$J(\zeta,\psi) = -\nabla\cdot(\vec{u}\zeta)$$

In order to define the flux on a unique control volume, we need to define the following quantities:



$$\overline{a}^{b_x}_{i,j+1/2} = \frac{1}{2}[(1-b)a_{i+1/2,j+1/2} + (1+b)a_{i-1/2,j+1/2}]$$

$$(u_b)_{i+1/2,j+1/2} = -\frac{1}{d}\left[\frac{(1-b)\psi_{i+1,j+1} + (1+b)\psi_{i,j+1}}{2} - \frac{(1-b)\psi_{i+1,j} + (1+b)\psi_{i,j}}{2}\right]$$

$$(v_b)_{i+1/2,j+1/2} = \frac{1}{d}\left[\frac{(1-b)\psi_{i+1,j+1} + (1+b)\psi_{i+1,j}}{2} - \frac{(1-b)\psi_{i,j+1} + (1+b)\psi_{i,j}}{2}\right]$$

The flux is then uniquely defined as $[\widehat{u\zeta}^{x,y,b}] = 2\gamma\overline{u_b}^y\overline{\zeta}^x + \gamma\overline{\overline{u}^x\widetilde{\zeta}}^{b_y}$, where $\gamma = 1/3$.

We obtain then the new general scheme on a unique control volume in a conservative form. Moreover it turns out that the parameter b acts as a *weight*. When the parameter is zero, we can read Arakawa's Jacobian in a new conservative form.

$$\nabla\cdot(\vec{u}\zeta) = [u\zeta]_x + [v\zeta]_y = [\widehat{u\zeta}^{x,y,b}]^{i+1/2,j}_{i-1/2,j} + [\widehat{v\zeta}^{y,x,b}]^{i,j+1/2}_{i,j-1/2}$$

Accuracy of the scheme

$$
\begin{aligned}
T(J_b) &= \psi_y\zeta_x - \psi_x\zeta_y \\
&+\frac{dx^2}{6}(\psi_y\zeta_{xxx} - \psi_x\zeta_{xxy} - \psi_{xx}\zeta_{xy} + \psi_{xy}\zeta_{xx} - \psi_{xxx}\zeta_y + \psi_{xxy}\zeta_x - \psi_x\zeta_{yyy} \\
&+\psi_y\zeta_{yyx} - \psi_{xy}\zeta_{yy} + \psi_{yy}\zeta_{xy} - \psi_{yyx}\zeta_y + \psi_{yyy}\zeta_x \\
&+\frac{b}{2}(2\psi_x\zeta_{yyx} - 2\psi_y\zeta_{xxy} + \psi_{xx}\zeta_{yy} - \psi_{yy}\zeta_{xx} + 2\psi_{xxy}\zeta_y - 2\psi_{yyx}\zeta_x)) \\
&+\frac{dx^4}{72}(\psi_{xy}\zeta_{xxxx} - 2\psi_{xx}\zeta_{xxxy} - 2\psi_{xxx}\zeta_{xxy} + 2\psi_{xxy}\zeta_{xxx} - \psi_{xxxx}\zeta_{xy} + 2\psi_{xxxy}\zeta_{xx} \\
&-2\psi_{xx}\zeta_{yyyx} + 2\psi_{yy}\zeta_{xxxy} - 2\psi_{xxx}\zeta_{yyy} + 2\psi_{yyy}\zeta_{xxx} - 2\psi_{xxxy}\zeta_{yy} + 2\psi_{yyyx}\zeta_{xx} \\
&-\psi_{xy}\zeta_{yyyy} + 2\psi_{yy}\zeta_{yyyx} - 2\psi_{yyx}\zeta_{yyy} + 2\psi_{yyy}\zeta_{yyx} - 2\psi_{yyyx}\zeta_{yy} + \psi_{yyyy}\zeta_{xy} \\
&+\frac{b}{2}(6\psi_{xx}\zeta_{xxyy} - \psi_{yy}\zeta_{xxxx} + 4\psi_{xxx}\zeta_{yyx} - 4\psi_{yyx}\zeta_{xxx} + \psi_{xxxx}\zeta_{yy} - 6\psi_{xxyy}\zeta_{xx} \\
&+\psi_{xx}\zeta_{yyyy} - 6\psi_{yy}\zeta_{xxyy} + 4\psi_{xxy}\zeta_{yyy} - 4\psi_{yyy}\zeta_{xxy} + 6\psi_{xxyy}\zeta_{yy} - \psi_{yyyy}\zeta_{xx})) + O(dx^6)
\end{aligned}
$$

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&+\psi_y\zeta_{yyx} - \psi_{xy}\zeta_{yy} + \psi_{yy}\zeta_{xy} - \psi_{yyx}\zeta_y + \psi_{yyy}\zeta_x \\
&+\frac{b}{2}(2\psi_x\zeta_{yyx} - 2\psi_y\zeta_{xxy} + \psi_{xx}\zeta_{yy} - \psi_{yy}\zeta_{xx} + 2\psi_{xxy}\zeta_y - 2\psi_{yyx}\zeta_x)) \\
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\end{aligned}
$$

→ a possible choice of b is the one that nullify the error proportional to dx^2

The modified wave number

Let's consider the special case

$$\psi(x, y) = -Uy + Vx$$

$$\zeta(x, y) = \overline{\zeta} e^{(i(mx+ny))}$$

with U, V, $\overline{\zeta}$ all constants. We get the analytic Jacobian:

$$J(\zeta, \psi) = -i(mu + nv)\zeta(x, y)$$

whilst the numerical Jacobian J_b (where $dx = dy$, $mx \rightarrow mjdx = j\alpha$, $ny \rightarrow nldx = l\beta$):

$$J_b(\zeta, \psi) = i\frac{\zeta(x, y)}{3dx}[-v(2sin(\beta) + sin(\beta)cos(\alpha)) - u(2sin(\alpha) + sin(\alpha)cos(\beta)) \\ +b(v(-sin(\alpha) + sin(\alpha)cos(\beta)) - u(sin(\beta) - sin(\beta)cos(\alpha)))]$$

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- b=0
  $J_0(\zeta,\psi) = i\frac{\zeta(x,y)}{3dx}[-v(2sin(\beta)+sin(\beta)cos(\alpha)) - u(2sin(\alpha)+sin(\alpha)cos(\beta))]$

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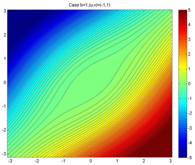
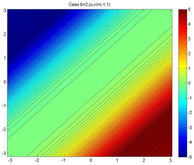
  $J_0(\zeta,\psi) = i\frac{\zeta(x,y)}{3dx}[-v(2sin(\beta)+sin(\beta)cos(\alpha)) - u(2sin(\alpha)+sin(\alpha)cos(\beta))]$
- $b = \pm 1$ and $u = \pm v$
  $J_{\pm 1}(\zeta,\psi) = i\frac{\zeta(x,y)}{3dx}v[-3sin(\beta) - 3sin(\alpha)]$

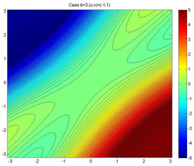
An example: the case $(U, V) = (-1, 1)$



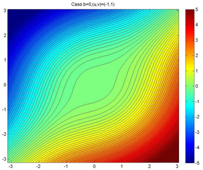
b=1



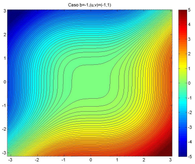
b=2



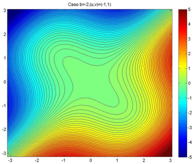
b=3



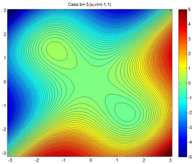
b=0



b=-1



b=-2

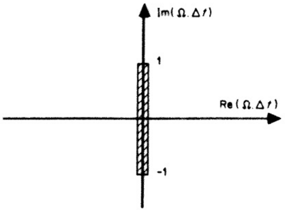


b=-3

How to fix the CFL condition with the free parameter

Let $R = \frac{u}{v}$, by the previous analysis on the mwn, we can estimate:

$$\begin{aligned} |\Omega| &\le |\frac{v}{3dx}|[2+1+|R|(2+1)+|b|(1+1+|R|(1+1))] \\ &= |\frac{v}{3dx}|[3+3|R|+|b|(2+2|R|)] \\ &= |\frac{v}{3dx}|[(3+2|b|)(1+|R|)] \end{aligned} \tag{4}$$



In order to get stability for the leap-frog scheme, we require:

$$\Delta t \le \frac{3dx}{max|v|}[(3+2|b|)(1+|R|)]^{-1} \tag{5}$$

in the case b=0 (Arakawa) we get the classical CFL condition:

$$\Delta t \le \frac{dx}{max|v| + max|u|}. \tag{6}$$

The advection equation

The advection equation

$$\frac{\partial \zeta}{\partial t} + \vec{u} \cdot \vec{\nabla} \zeta = 0$$

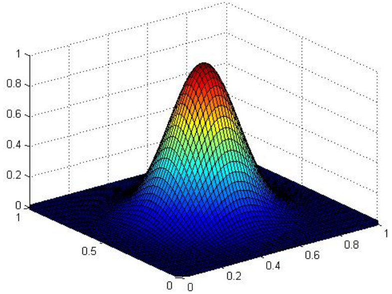
The advection equation

The advection equation

$$\frac{\partial \zeta}{\partial t} + \vec{u} \cdot \vec{\nabla} \zeta = 0$$

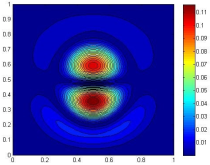
Time discretization at time N:

- if N=1: Eulero scheme
- if ($N > 1$ .AND. mod(N,10)=0): Matzuno scheme
- else: Leap-Frog scheme

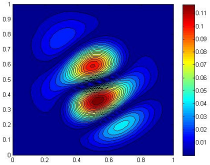


Results: $(u, v) = (1, 0)$

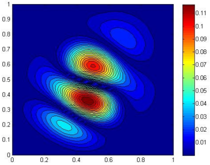
b	**T**	**N**	**CFL**	L^∞	L^1	L^2
0	5	64	0.5	1.2173 e-001	4.2880 e-002	2.5707 e-002
1	5	64	0.5	1.2261 e-001	4.2684 e-002	2.6346 e-002
-1	5	64	0.5	1.2261 e-001	4.2684 e-002	2.6346 e-002



b=0



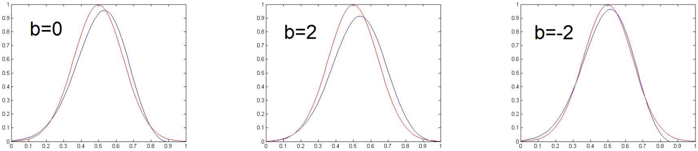
b=1



b=-1

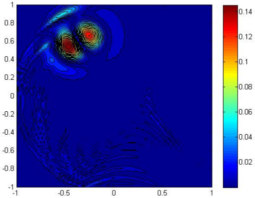
Results: $(u, v) = (-1, 1)$

b	T	N	CFL	L^∞	L^1	L^2
-3	5	64	0.5	1.0775 e-001	4.0348 e-002	3.0349 e-002
-2	5	64	0.5	1.0267 e-001	3.8392 e-002	2.5415 e-002
-1	5	64	0.5	1.3730 e-001	4.0045 e-002	2.7721 e-002
0	5	64	0.5	1.8564 e-001	4.6229 e-002	3.4902 e-002
1	5	64	0.5	2.2337 e-001	5.8602 e-002	4.3846 e-002
2	5	64	0.5	2.4628 e-001	7.1161 e-002	5.2478 e-002
3	5	64	0.5	2.5960 e-001	8.2244 e-002	6.1007 e-002

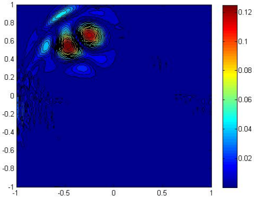


Results: $(u, v) = (-\omega y, \omega x)$

b	T	Δt	N	L^∞	L^1	L^2
0	5	1.0101e-002	100	1.5263e-001	5.0805e-002	3.0419e-002
0	5	6.0000e-003	100	1.6813e-001	5.4615e-002	3.3490e-002
$b = \frac{u^2+v^2}{2uv}$, $\lvert b \rvert \leq 1$	5	6.0606e-003	100	1.3879e-001	4.4813e-002	2.7747e-002
$b = \frac{u^2+v^2}{2uv}$, $\lvert b \rvert \leq 2$	5	4.3290e-003	100	1.3245e-001	4.1187e-002	2.6672e-002
$b = \frac{u^2+v^2}{2uv}$, $\lvert b \rvert \leq 3$	5	3.3670e-003	100	1.3139e-001	4.1407e-002	2.6526e-002
$b = \frac{u^2+v^2}{2uv}$, $\lvert b \rvert \leq 4$	5	2.7548e-003	100	1.3106e-001	4.0865e-002	2.6511e-002
$b = \frac{u^2+v^2}{2uv}$, $\lvert b \rvert \leq 5$	5	2.3310e-003	100	1.3090e-001	4.0712e-002	2.6554e-002
$b = \frac{u^2+v^2}{2uv}$, $\lvert b \rvert \leq 6$	5	2.0202e-003	100	1.3079e-001	4.0657e-002	2.6608e-002



[b=0]



[$|b| < 4$]

The Rankine model

The Rankine vortex model is a circular flow in which an inner circular region about the origin is in solid rotation, while the outer region is free of vorticity. Its definition is natural in a cylindrical coordinate system $(r; \theta; z)$:

$$\begin{cases} v_r & = & 0 \\ v_\theta = \begin{cases} V_R \frac{r}{R} & \textit{if } 0 \leq r < R \\ V_R \frac{R}{r} & \textit{if } R \leq r \end{cases} \\ v_z & = & 0 \end{cases}$$

$$\vec{\zeta} = \nabla \times \vec{v} = \hat{k} \frac{1}{r} \frac{\partial (r v_\theta)}{\partial r} = \hat{k} (\frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r})$$

$$= \begin{cases} 2 \, \frac{V_R}{R} & \text{if } 0 \leq r < R \\ 0 & \text{if } R \leq r \end{cases}$$

It is worth to note that the Rankine vortex is characterized by a continuous velocity field, but with a discontinuity in vorticity at the characteristic distance, and it is well known that this will cause problems in numerical simulations.

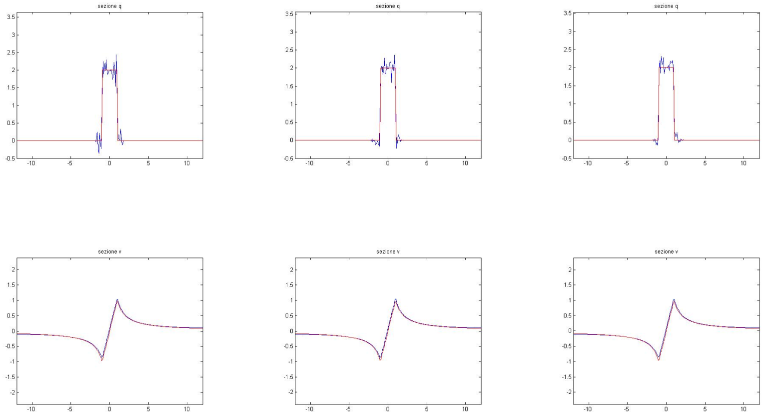
Results

$$\begin{cases} \zeta_t = J(\zeta(\vec{x},t),\psi(\vec{x},t)) \\ \zeta = \Delta\psi \\ \text{(solved by Gauss-Seidel algorithm via Multi-Grid method)} \\ \\ \zeta(\vec{x},0) = \zeta_0(\vec{x}) \\ \zeta(\vec{x},t) = 0 \quad \text{if } \vec{x} \in \partial D \\ \psi(\vec{x},t) = B(\vec{x},t) \quad \text{if } \vec{x} \in \partial D \\ \text{-constant velocity at the boundary-} \end{cases}$$

- CFL=1
- Domain: [-12,12]x[-12,12]
- Nodes NxN=257x257
- Multigrid Levels LL=3
- T=3

Test n°	**B**	L^∞	L^1	L^2
1a	0	9.9394e-001	8.4248e-001	5.5692e-001
1b	1	1.0726e+000	8.4771e+001	5.7087e-001
1c	-1	1.0728e-000	8.9904e-001	5.7087e-001
1d	2	1.0198e-000	8.8891e-001	5.9787e-001
1e	-2	1.0200-000	9.6736e-001	5.9788e-001
1f	average, $\|b\| < 1$	1.2594e+000	1.0919e+000	6.5639e-001
1g	-average, $\|b\| < 1$	8.6986e-001	7.4244e-001	4.8888e-001

Results



Concluding remarks

- General finite differences discretization of a general operator which uses a 9x9 grid nodes
- Forcing the scheme to:
  - be skew-symmetric
  - conserve energy
  - conserve enstrophy
  - be consistent at order two

  we obtain a scheme with one free parameter. With the parameter b=0 we recover Arakawa's solution.

  1. b as weight for calculating derivatives (meaning velocities)
  2. Study of the parameter in terms of dispersion error
     - Truncation error
     - Modified Wave Number
- Reformulation in terms of Lax-Wendroff incoming and outgoing fluxes $\rightarrow$ New Standard Conservative formulation of Arakawa's Jacobian and of the general scheme

Other works

- Rick Salmon, Lynne D. Talley, *Generalizations of Arakawa's Jacobian*. Journal of Computational Physics, 01/1989; 83(2):247-259
- Rick Salmon, *A General Method for Conserving Energy and Potential Enstrophy in Shallow-Water Models*. J. Atmos. Sci., 64, 515531 (2006)
- Robert I. McLachlan, *Spatial discretization of partial differential equations with integrals*. IFS, Massey University, Palmerston North, New Zealand (1998)
- A. Arakawa, V. Lamb, *A potential Enstrophy and Energy Conserving Scheme for the Shallow Water Equations*. Monthly Weather Review, vol. 109, Jan. 1981, p. 18-36. U.S. Environmental Protection Agency
- Andrew T. T. McRae and Colin J. Cotter, *Energy -and enstrophy- conserving schemes for the shallow-water equations, based on mimetic finite elements*. Quarterly Journal of the Royal Meteorological Society (2013)

Next steps

- Extension to different hydro-dynamic systems
- Non-linear criterion to choose effectively the best parameter
- Higher order generalized scheme
- Application to a quasi-geostrophic system (joint work with ENEA)

Thank you for your attention.