An energy-conserving quasi-hydrostatic deep-atmosphere dynamical core

Marine Tort[1], Thomas Dubos[1] and Thomas Melvin[2]

marine.tort@lmd.polytechnique.fr
www.lmd.jussieu.fr/∼mtlmd

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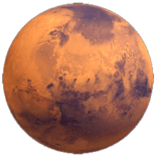
PDEs on the Sphere, NCAR, Co

[1] *Laboratoire de Météorologie Dynamique, Ecole Polytechnique, France*
[2] *Met Office, Exeter, UK*

Motivations : LMD-Z and Planets

Atmospheric component of the IPSL model

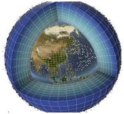
Planetary atmosphere version
e.g. Mars



Common dynamical core on a longitude/latitude grid :

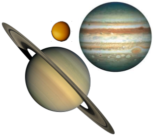
- *shallow-atmosphere* hydrostatic equations (HPE),
- enstrophy-conserving scheme (*Sadourny, 1975a*).

Motivations : LMD-Z and Planets



Atmospheric component of the IPSL model

Planetary atmosphere version
e.g. Giant gas planets, Titan
$\rightarrow$ DEEP ATMOSPHERES



The dynamical core requires a *deep-atmosphere* version.

GOALS

- solve the *deep-atmosphere* quasi-hydrostatic equations (QHE) (*White and Bromley, 1995*) AND the recently derived non-traditional *shallow-atmosphere* equations (NTE) with complete Coriolis force (*Tort and Dubos, 2014a*),
- preserve some discrete conservation properties.

An energy-conserving scheme (Lagrangian vertical coordinate case)
- Curl-form and Hamiltonian formulation
- How to conserve energy ?
- Iterative procedure to solve hydrostasy

Application to LMD-Z
- Mass coordinate : how is that different from a Lagrangian vertical coordinate ?
- Idealized test cases

Summary

Curl-form and Hamiltonian formulation

Derivation in curl-form using a time-dependent curvilinear coordinates system

- from White and Bromley (1995)'s equations (spherical coordinates and advective form),
- using a general vertical coordinate η : $r(\lambda, \phi, \eta; t)$.

Momentum prognostic variables $\tilde{\mathbf{u}} = (\tilde{u}, \tilde{v})$ (*Dubos and Tort, 2014*)

- *shallow-atmosphere* $\tilde{\mathbf{u}} = (r_0 \cos\phi(u + \Omega r_0 \cos\phi), r_0 v)$,
- *deep-atmosphere* $\tilde{\mathbf{u}} = (r \cos\phi(u + \Omega r \cos\phi), rv)$,

Momentum

$$\partial_t \tilde{\mathbf{u}} + \frac{1}{\tilde{\rho}} (\nabla \times \tilde{\mathbf{u}}) \times \mathbf{U} + \nabla (K + \Phi) + \theta \nabla \pi = 0$$

Mass

$$\tilde{\rho} = \rho r^2 \cos\phi \partial_\eta r, \quad \partial_t \tilde{\rho} + \nabla \cdot \mathbf{U} = 0$$

Entropy

$$\Theta = \tilde{\rho}\theta, \quad \partial_t \Theta + \nabla \cdot (\theta \mathbf{U}) = 0$$

Curl-form and Hamiltonian formulation

Variational interpretation

- Hamiltonian $\mathcal{H}(\tilde{\rho}, \tilde{\mathbf{u}}, \Theta, r) = \int_{\mathcal{V}} \mathrm{d}\lambda \mathrm{d}\phi \mathrm{d}\eta \; \tilde{\rho} \left(K(\tilde{\mathbf{u}}, r) + \Phi(r) + e\left(\frac{\cos\phi \partial_\eta \tilde{r}^3}{3\tilde{\rho}}, \frac{\Theta}{\tilde{\rho}} \right) \right)$
- equations are written in term of functional derivatives of $\mathcal{H}$

Functional derivatives

$$\triangleright \; \frac{\delta \mathcal{H}}{\delta \tilde{\rho}} = K + \Phi \qquad \triangleright \; \frac{\delta \mathcal{H}}{\delta \tilde{\mathbf{u}}} = \mathbf{U} \qquad \triangleright \; \frac{\delta \mathcal{H}}{\delta \Theta} = \pi$$

$$\triangleright \; \frac{\delta \mathcal{H}}{\delta r} = -\tilde{\rho} \left(\frac{u^2 + v^2}{r} + 2\Omega \cos\phi u \right) + r^2 \cos\phi \partial_\eta p + \tilde{\rho} g(r)$$

HYDROSTATIC CONSTRAINT : $\frac{\delta \mathcal{H}}{\delta r} = 0$

How to conserve energy ?

Imitate the exact Hamiltonian formulation at discrete level

Salmon, 2004

1. Choose the spatial grid

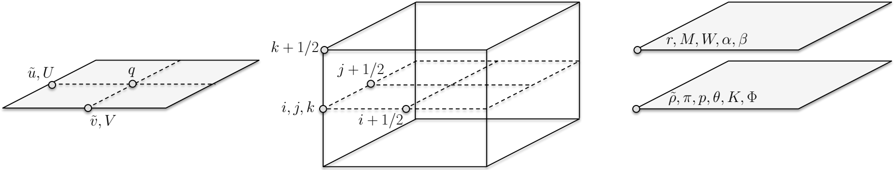


FIG 1. *Grid straggering*

- horizontal : C-grid,
- vertical : Lorenz grid.

How to conserve energy ?

2. Express the discrete energy budget, take care that the terms compensate.

$$\begin{aligned}
\frac{d\mathcal{H}}{dt} &= \sum \left[\frac{\delta\mathcal{H}}{\delta\tilde{\rho}}\partial_t\tilde{\rho} + \frac{\delta\mathcal{H}}{\delta\Theta}\partial_t\Theta + \frac{\delta\mathcal{H}}{\delta\tilde{u}}\partial_t\tilde{u} + \frac{\delta\mathcal{H}}{\delta\tilde{v}}\partial_t\tilde{v} + \frac{\delta\mathcal{H}}{\delta r}\partial_t r \right], \\
&= -\sum \left[\frac{\delta\mathcal{H}}{\delta\tilde{\rho}}\left(\delta_i\frac{\delta\mathcal{H}}{\delta\tilde{u}} + \delta_j\frac{\delta\mathcal{H}}{\delta\tilde{v}}\right) + \frac{\delta\mathcal{H}}{\delta\Theta}\left(\delta_i\left(\overline{\theta}^i\frac{\delta\mathcal{H}}{\delta\tilde{u}}\right) + \delta_j\left(\overline{\theta}^j\frac{\delta\mathcal{H}}{\delta\tilde{v}}\right)\right)\right. \\
&\quad + \frac{\delta\mathcal{H}}{\delta\tilde{u}}\left(-\overline{\frac{\delta_i\tilde{v}-\delta_j\tilde{u}}{\overline{\tilde{\rho}}^{ij}}\overline{\frac{\delta\mathcal{H}}{\delta\tilde{v}}}^{i}}^{j} + \delta_i\frac{\delta\mathcal{H}}{\delta\tilde{\rho}} + \overline{\theta}^i\delta_i\frac{\delta\mathcal{H}}{\delta\Theta}\right) \\
&\quad \left. + \frac{\delta\mathcal{H}}{\delta\tilde{v}}\left(\overline{\frac{\delta_i\tilde{v}-\delta_j\tilde{u}}{\overline{\tilde{\rho}}^{ij}}\overline{\frac{\delta\mathcal{H}}{\delta\tilde{u}}}^{j}}^{i} + \delta_j\frac{\delta\mathcal{H}}{\delta\tilde{\rho}} + \overline{\theta}^j\delta_j\frac{\delta\mathcal{H}}{\delta\Theta}\right) - \frac{\delta\mathcal{H}}{\delta r}\partial_t r\right] = 0\ !
\end{aligned}$$

3. Discretize the Hamiltonian and deduce the discrete derivatives

$$\mathcal{H} = \sum_{\tilde{\rho}}\tilde{\rho}\left(\frac{1}{2}\overline{\left(\frac{\tilde{u}}{\overline{r}^{ik}} - \Omega\overline{r}^{ik}\cos\phi\right)^2}^{i} + \frac{1}{2}\overline{\left(\frac{\tilde{v}}{\overline{r}^{jk}}\right)^2}^{j} + e\left(\frac{\cos\phi\,\delta_k r^3}{3\tilde{\rho}}, \frac{\Theta}{\tilde{\rho}}\right) + \Phi(\overline{r}^k)\right)$$

Iterative procedure to solve hydrostasy

HPE	QHE
$\delta_k p = -\overline{\tilde{\rho}}^k g$	$\delta_k p = -\overline{\tilde{\rho} g(\overline{r}^k)}^k + \overline{\overline{\tilde{\rho}}^i \left(\frac{\tilde{u}^2}{\left(\overline{r}^{ik}\right)^3 \cos^2\phi} + \Omega^2 \overline{r}^{ik} \cos\phi \right)}^{ik} + \overline{\overline{\tilde{\rho}}^j \left(\frac{\tilde{v}^2}{\left(\overline{r}^{jk}\right)^3} \right)}^{jk}$
$a^2 \delta_k r = \frac{\tilde{\rho}\kappa\theta c_p p^{\kappa-1}}{p_r^\kappa \cos\phi}$	$\delta_k r^3 = \frac{3\tilde{\rho}\kappa\theta c_p p^{\kappa-1}}{p_r^\kappa \cos\phi}$

r is the solution of a non-linear elliptic problem whose p is a byproduct.

- HPE : direct obvious solution,
- QHE : iterative solution (fix point or Newton's method).

Mass coordinate : how is that different from a Lagrangian vertical coordinate ?

Mass budget : $\partial_t \tilde{\rho} + \partial_\lambda U + \partial_\phi V + \partial_\eta (\tilde{\rho}\dot{\eta})$

- $\tilde{\rho}$ is not a prognostic variable anymore and $\mathcal{H}$ is expressed with respect to integrated mass M_s instead of local mass $\tilde{\rho}$,
- $\dot{\eta} \neq 0$, there is additional non-zero vertical transport in the equations.

Vertical relabeling symmetry (*Dubos and Tort, 2014*)

- compensation of vertical transports terms due to vertical relabeling symmetry,
- we may imitate to the discrete level this relabeling to cancel vertical transport in the discrete energy budget (*Tort et al, in preparation*).

Idealized test cases

Like-Earth planet experiment

Baroclinic instability
Ullrich et al, 2013

Newtonian relaxation
Held and Suarez, 1994

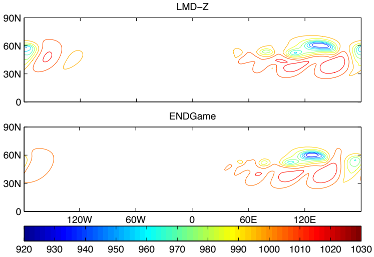


FIG 4. *Surface pressure p_s at day 10*

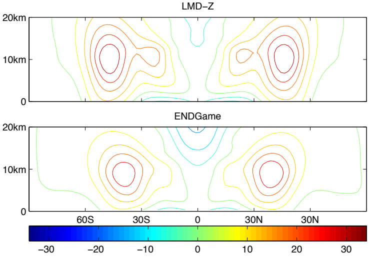


FIG 5. *Zonally averaged zonal velocity over 1000 planetary rotations*

Idealized test cases

Small like-Earth planet experiment

Non-traditional regime
$X \sim U/(\Omega_e H) \Rightarrow X = 15$

Deep-atmosphere regime
$X \gg U/(\Omega_e H)$ e.g $X = 50$

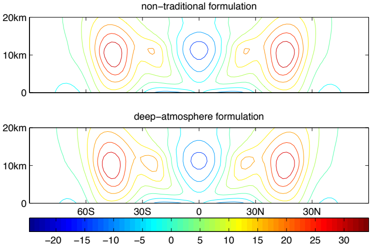
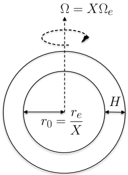


FIG 6. *Zonally averaged zonal velocity over 1000 planetary rotations* $X = 15$



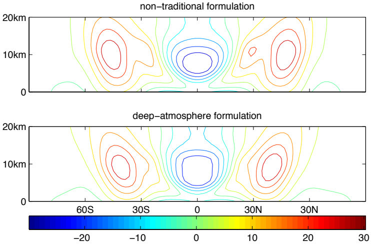


FIG 7. *Zonally averaged zonal velocity over 1000 planetary rotations* $X = 50$

development of a zonally averaged easterly flow in the tropics scaled by $U = -2X\Omega_0 H \cos\phi$ (*White and Bromley, 1995, Wedi and Smolarkiewicz, 2009*)

Summary

Implementation of the QHE into LMD-Z

- energy-conserving,
- systematic method to discretize hydrostatic systems,
- ongoing : newtonian relaxation on a Titan-like planet.

Tort M. and Dubos T., 2014a Dynamically consistent shallow-atmosphere equations with a complete Coriolis force. *Q. J. R. Meteor. Soc. (in press, early view)*

Tort M. and Dubos T., 2014b. Usual approximations to the equations of atmospheric motion : a variational perspective. *J. Atmos. Sci. (accepted)*

Tort M. et al, 2014. Consistent shallow-water equations on the rotating sphere with complete Coriolis force and topography. *J. Fluid Mech. (accepted)*

Dubos T. and Tort M., 2014. Equations of atmospheric motion in non-Eulerian vertical coordinates : vector-invariant form and Hamiltonian formulation. *Mont. Weath. Rev. (submitted)*

Consequence of vertical relabeling

Energy budget induced by vertical flux (mass-based coordinate) $W = \tilde{\rho}\dot{\eta}$

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} &= \sum \left[\frac{\delta\mathcal{H}}{\delta\tilde{\rho}}\partial_t\tilde{\rho} + \frac{\delta\mathcal{H}}{\delta\Theta}\partial_t\Theta + \frac{\delta\mathcal{H}}{\delta\tilde{u}}\partial_t\tilde{u} + \frac{\delta\mathcal{H}}{\delta\tilde{v}}\partial_t\tilde{v} + \frac{\delta\mathcal{H}}{\delta r}\partial_t r\right], \\
&= -\sum \left[\frac{\delta\mathcal{H}}{\delta\tilde{\rho}}\left(\delta_i\frac{\delta\mathcal{H}}{\delta\tilde{u}} + \delta_j\frac{\delta\mathcal{H}}{\delta\tilde{v}} + \delta_k W\right) + \frac{\delta\mathcal{H}}{\delta\Theta}\left(\delta_i\left(\overline{\theta}^i\frac{\delta\mathcal{H}}{\delta\tilde{u}}\right) + \delta_j\left(\overline{\theta}^j\frac{\delta\mathcal{H}}{\delta\tilde{v}}\right) + \delta_k\left(\overline{\theta}^k W\right)\right)\right. \\
&\quad + \frac{\delta\mathcal{H}}{\delta\tilde{u}}\left(\frac{\overline{\overline{W}^i\delta_k\tilde{u}}^k}{\overline{\tilde{\rho}}^i} - \overline{\frac{\delta_i\tilde{v}-\delta_j\tilde{u}}{\overline{\tilde{\rho}}^{ij}}\overline{\frac{\delta\mathcal{H}}{\delta\tilde{v}}}^i}^j + \delta_i\frac{\delta\mathcal{H}}{\delta\tilde{\rho}} + \overline{\theta}^i\delta_i\frac{\delta\mathcal{H}}{\delta\Theta}\right) \\
&\quad \left. + \frac{\delta\mathcal{H}}{\delta\tilde{v}}\left(\frac{\overline{\overline{W}^j\delta_k\tilde{v}}^k}{\overline{\tilde{\rho}}^j} + \overline{\frac{\delta_i\tilde{v}-\delta_j\tilde{u}}{\overline{\tilde{\rho}}^{ij}}\overline{\frac{\delta\mathcal{H}}{\delta\tilde{u}}}^j}^i + \delta_j\frac{\delta\mathcal{H}}{\delta\tilde{\rho}} + \overline{\theta}^j\delta_j\frac{\delta\mathcal{H}}{\delta\Theta}\right) - \frac{\delta\mathcal{H}}{\delta r}\partial_t r\right] = 0\ ?
\end{aligned}$$

How to cancel the blue terms ? Bernoulli function $B = \frac{\delta\mathcal{H}}{\delta\tilde{\rho}}$ is such as :

$$\begin{aligned}
\delta_k B + \overline{\theta}^k\delta_k\frac{\delta\mathcal{H}}{\delta\Theta} - \overline{\overline{\frac{1}{\overline{\tilde{\rho}}^i}\frac{\delta\mathcal{H}}{\delta\tilde{u}}}^k\delta_k\tilde{u}}^i - \overline{\overline{\frac{1}{\overline{\tilde{\rho}}^j}\frac{\delta\mathcal{H}}{\delta\tilde{v}}}^k\delta_k\tilde{v}}^j &= 0 \\
\sum_k \delta_k\beta B &= \sum_k \delta_k\beta(K+\Phi)
\end{aligned}$$

Consequence of vertical relabeling

Tort and Dubos, 2014a

$$\frac{Du}{Dt} - \left(2\Omega\left(1+\frac{2z}{r_0}\right) + \frac{u}{r_0\cos\phi}\right) v\sin\phi + 2\Omega\cos\phi w + \frac{1}{\rho r_0\cos\phi}\frac{\partial p}{\partial\lambda} = 0$$

$$\frac{Dv}{Dt} + \left(2\Omega\left(1+\frac{2z}{r_0}\right) + \frac{u}{r_0\cos\phi}\right) u\sin\phi + \frac{1}{\rho r_0}\frac{\partial p}{\partial\phi} = 0$$

$$\delta_{NH}\frac{Dw}{Dt} + 2\Omega\cos\phi u + g + \frac{1}{\rho}\frac{\partial p}{\partial z} = 0$$

Non-traditional shallow-atmosphere angular momentum is now conserved

$$a\cos\phi\left(u + \Omega r_0\cos\phi\left(r_0 + 2z\right)\right)$$

Stability analysis of the vertical discretization

Isothermal atmosphere at rest on the $f - F$-plane - free pressure surface BC
extension of *Thuburn et al, 2002b*'s work with a rigid lid BC.

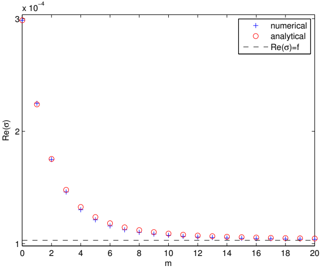


FIG 2. *Analytical vs numerical frequency spectrum*

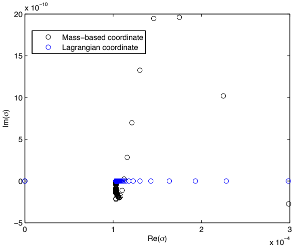


FIG 3. *Numerical eigenvalues with a Lagrangian vs mass-based coordinate*

- Lagrangian coordinate $1/\sigma \to \infty$,
- Mass-based vertical coordinate $1/\sigma \to 15$ years.