An accurate and efficient framework for adaptive numerical weather prediction

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Boulder, 9 April 2014

Outline

- Motivation and overview
- What's new on SISLDG formulation:
  - a novel SISL time integration approach
  - from the horizontal towards the vertical discretization
  - p-reduced "grid"
- Numerical validation:
  - horizontal:
    - Williamson's test 5
    - Williamson's test 6
  - vertical (preliminary results):
    - acoustic wave propagation
    - Warm bubble test
    - Inertia-gravity wave test
- Conclusions and future plans

- In collaboration with Luca Bonaventura (MOX-Politecnico di Milano)
- with thanks to Filippo Giorgi (ICTP Abdus Salam Trieste)
- and acknowledgements for funding from
  - The Abdus Salam International Centre for Theoretical Physics
  - The INdAM-GNCS
  - The Italian Ministery of Research and Education

Overview

- Goal: design a new generation *nonhydrostatic* dynamical core for the *regional* climate modelling system RegCM, developed at Abdus Salam ICTP-Trieste.

- First step: p-SISLDG, G.Tumolo, L.Bonaventura, M.Restelli, JCP 2013
  - coupling DG to SI-SL techniques (no CFL conditions);
  - introduction of p-adaptivity (flexible degrees of freedom);

- then extended to spherical geometry.

- Open issues:
  - first order accurate in time if off-centering was used;
  - pole problem (not for stability but for efficiency);
  - vertical discretization.

A novel SISL time integration approach: TR-BDF2

Given a Cauchy problem

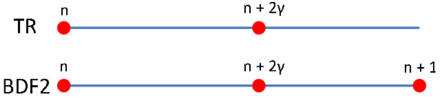
$$\begin{aligned} \mathbf{y}' &= \mathbf{f}(\mathbf{y}, t) \\ \mathbf{y}(0) &= \mathbf{y}_0 \end{aligned}$$

the TR-BDF2 method is defined by the two following implicit stages (Bank et al. IEEE trans. 1985):

$$\begin{aligned} \mathbf{u}^{n+2\gamma} - \gamma \Delta t \mathbf{f}(\mathbf{u}^{n+2\gamma}, t_n + 2\gamma\Delta t) &= \mathbf{u}^n + \gamma \Delta t \mathbf{f}(\mathbf{u}^n, t_n), \\ \mathbf{u}^{n+1} - \gamma_2 \Delta t \mathbf{f}(\mathbf{u}^{n+1}, t_{n+1}) &= (1-\gamma_3)\mathbf{u}^n + \gamma_3 \mathbf{u}^{n+2\gamma}. \end{aligned}$$

with $\gamma \in [0, 1/2]$ fixed implicitness parameter and

$$\gamma_2 = \frac{1-2\gamma}{2(1-\gamma)}, \quad \gamma_3 = \frac{1-\gamma_2}{2\gamma}.$$



Advantages of TR-BDF2

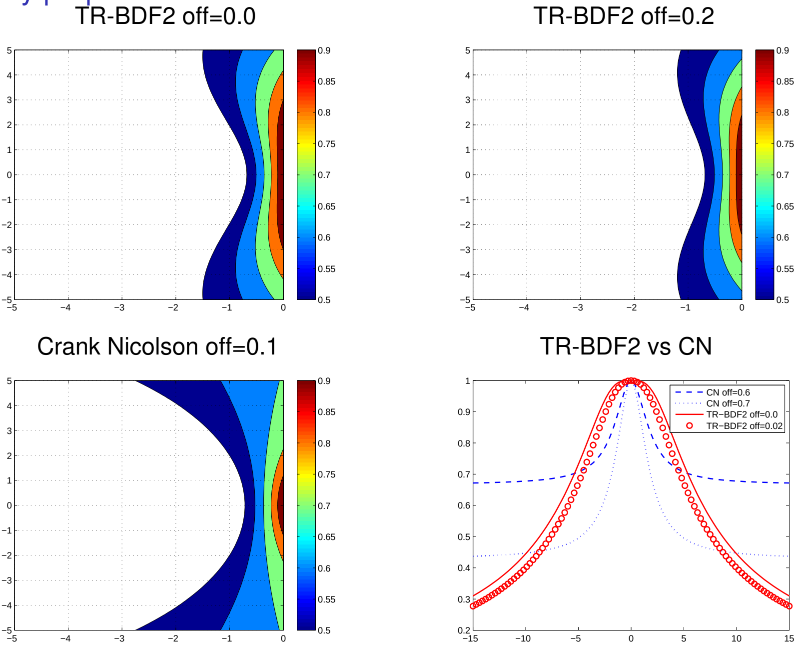
TR-BDF2, reformulated as a SDIRK method (Hosea Shampine ANM 1996),

$$
\begin{aligned}
\mathbf{k}_1 &= \mathbf{f}\left(\mathbf{u}^n, t_n\right), \\
\mathbf{k}_2 &= \mathbf{f}\left(\mathbf{u}^n + \gamma \Delta t \mathbf{k}_1 + \gamma \Delta t \mathbf{k}_2, t_n + \gamma \Delta t\right), \\
\mathbf{k}_3 &= \mathbf{f}\left(\mathbf{u}^n + \frac{1-\gamma}{2} \Delta t \mathbf{k}_1 + \frac{1-\gamma}{2} \Delta t \mathbf{k}_2 + \gamma \Delta t \mathbf{k}_3, t_{n+1}\right), \\
\mathbf{u}^{n+1} &= \mathbf{u}^n + \Delta t \left(\frac{1-\gamma}{2} \mathbf{k}_1 + \frac{1-\gamma}{2} \mathbf{k}_2 + \gamma \mathbf{k}_3\right),
\end{aligned}
$$

exhibits interesting properties like:

- it is L-stable;
- it is second order accurate but embedded in a third order companion (hence “free" asymptotically correct error estimate);
- all the stages are evaluated within the the step interval;
- it is First-Same-As-Least (FSAL), hence only two implicit stages to evaluate per step;
- all the implicit stages have the same coefficients, hence they can be solved once.

Stability properties of TR-BDF2



SL reinterpretation of TR-BDF2: Shallow Water Equations (SWE)

- If viewed as a composite scheme, TRBDF2 can be easily reformulated in a semi-Lagrangian way, as time integrator for PDEs in advective form;

- if suitable semi-Lagrangian evolution operators for scalar end vector valued functions are introduced: $[E(t^i, \Delta\tau)G](\mathbf{x}) = G(t^i, \mathbf{x}_D)$ where $\mathbf{x}_D = \mathbf{x} - \int_{t^i}^{t^i+\Delta\tau} \mathbf{u}^n \left(\mathbf{X}(t; t^i + \Delta\tau, \mathbf{x})\right) dt$ and $\mathbf{X}(t^i; t^i + \Delta\tau, \mathbf{x})$ is the solution of:

$$\begin{cases} \frac{d}{dt}\mathbf{X}(t; t^i + \Delta\tau, \mathbf{x}) = \mathbf{u}^n \Big(\mathbf{X}(t^i; t^i + \Delta\tau, \mathbf{x})\Big) \\ \mathbf{X}(t^i + \Delta\tau; t^i + \Delta\tau, \mathbf{x}) = \mathbf{x} \end{cases} ,$$

- if Shallow Water Equations (SWE) are to be solved:

$$\frac{Dh}{Dt} = -h\nabla \cdot \mathbf{u},$$
$$\frac{D\mathbf{u}}{Dt} = -g\nabla h - f\hat{\mathbf{k}} \times \mathbf{u} - g\nabla b,$$

with $h, \mathbf{u}$ and b being fluid depth, velocity and bathymetry elevation respectively, and $\frac{D}{Dt}$ denoting the Lagrangian derivative operator,

SL reinterpretation of TR-BDF2: SWE

... then the TR stage of the SISL time semi-discretization of the SWE equations in vector form is given by

$$\begin{aligned} h^{n+2\gamma} \quad &+ \quad \gamma\Delta t \ h^n \ \nabla \cdot \mathbf{u}^{n+2\gamma} \\ &= \quad E\left(t^n, 2\gamma\Delta t\right) \left[h - \gamma\Delta t \ h \ \nabla \cdot \mathbf{u}\right], \end{aligned}$$

$$\mathbf{u}^{n+2\gamma} + \gamma\Delta t\Big[g\nabla h^{n+2\gamma} + f\hat{\mathbf{k}} \times \mathbf{u}^{n+2\gamma}\Big] = -\gamma\Delta t \ g\nabla b$$
$$+E(t^n, 2\gamma\Delta t)\left\{\mathbf{u} - \gamma\Delta t\left[g(\nabla h + \nabla b) + f\hat{\mathbf{k}} \times \mathbf{u}\right]\right\},$$

followed by the BDF2 stage:

$$\begin{aligned} h^{n+1} \quad &+ \quad \gamma_2\Delta t \ h^{n+2\gamma} \ \nabla \cdot \mathbf{u}^{n+1} \\ &= \quad (1-\gamma_3)E\left(t^n, \Delta t\right)h \\ &+ \quad \gamma_3 E\left(t^n + 2\gamma\Delta t, (1-2\gamma)\Delta t\right)h, \end{aligned}$$

$$\mathbf{u}^{n+1} + \gamma_2\Delta t\Big[g\nabla h^{n+1} + f\hat{\mathbf{k}} \times \mathbf{u}^{n+1}\Big] = -\gamma_2\Delta t \ g\nabla b$$
$$+(1-\gamma_3)E\left(t^n, \Delta t\right)\mathbf{u} + \gamma_3 E\left(t^n + 2\gamma\Delta t, (1-2\gamma)\Delta t\right)\mathbf{u}.$$

Vertical Slice Equations (VSE)

- Euler equations (neglecting Coriolis force) in terms of $\Theta = T\left(\frac{p}{p_0}\right)^{-R/c_p}, \Pi = \left(\frac{p}{p_0}\right)^{R/c_p}$:

$$\frac{D\Pi}{Dt} + \left(\frac{c_p}{c_v} - 1\right)\Pi\nabla\cdot\boldsymbol{u} = 0,$$
$$\frac{D\boldsymbol{u}}{Dt} + c_p\Theta\nabla\Pi = -g\boldsymbol{k},$$
$$\frac{D\Theta}{Dt} = 0,$$

being $\frac{D}{Dt}$ the Lagrangian derivative, c_p, c_v, R the constant pressure and constant volume specific heats and the gas constant of dry air.

- Decompose thermodynamic variables in basic state and perturbation:

$$\Pi(x, y, z, t) = \pi^*(z) + \pi(x, y, z, t)$$
$$\Theta(x, y, z, t) = \theta^*(z) + \theta(x, y, z, t)$$

where π^*, θ^* are chosen s.t. $\quad c_p\theta^*\frac{d\pi^*}{dz} = -g,$

- and consider a vertical slice ($\frac{\partial}{\partial y} = 0$):

$$\frac{D\Pi}{Dt} + \left(\frac{c_p}{c_v} - 1\right)\Pi\nabla\cdot\boldsymbol{u} = 0,$$
$$\frac{Du}{Dt} + c_p\Theta\frac{\partial\pi}{\partial x} = 0,$$
$$\frac{Dw}{Dt} + c_p\Theta\frac{\partial\pi}{\partial z} - g\frac{\theta}{\theta^*} = 0,$$
$$\frac{D\theta}{Dt} + w\frac{d\theta^*}{dz} = 0.$$

SL counterpart of TR stage for VSE

$$\pi^{n+2\gamma} + \gamma\Delta t \ \left(c_p/c_v - 1\right)\Pi^n \nabla\cdot\mathbf{u}^{n+2\gamma} = -\pi^* + E\left(t^n, 2\gamma\Delta t\right)\left[\Pi - \gamma\Delta t\left(c_p/c_v - 1\right)\Pi \ \nabla\cdot\mathbf{u}\right],$$

$$u^{n+2\gamma} + \gamma\Delta t \ c_p\Theta^n \frac{\partial\pi}{\partial x}^{n+2\gamma} = E(t^n, 2\gamma\Delta t)\left[u - \gamma\Delta t \ c_p\Theta\frac{\partial\pi}{\partial x}\right],$$

$$w^{n+2\gamma} + \gamma\Delta t\left(c_p\Theta^n\frac{\partial\pi}{\partial z}^{n+2\gamma} - g\frac{\theta^{n+2\gamma}}{\theta^*}\right) = E(t^n, 2\gamma\Delta t)\left[w - \gamma\Delta t\left(c_p\Theta\frac{\partial\pi}{\partial z} - g\frac{\theta}{\theta^*}\right)\right],$$

$$\theta^{n+2\gamma} + \gamma\Delta t\frac{d\theta^*}{dz}w^{n+2\gamma} = E(t^n, 2\gamma\Delta t)\left[\theta - \gamma\Delta t\frac{d\theta^*}{dz}w\right].$$

Inserting the discretized energy eq. into the discrete vertical momentum eq.:

$$\left(1 + (\gamma\Delta t)^2\frac{g}{\theta^*}\frac{d\theta^*}{dz}\right)w^{n+2\gamma} + \gamma\Delta t c_p\Theta^n\frac{\partial\pi}{\partial z}^{n+2\gamma} =$$
$$E(t^n, 2\gamma\Delta t)\left[w - \gamma\Delta t\left(c_p\Theta\frac{\partial\pi}{\partial z} - g\frac{\theta}{\theta^*}\right)\right] + \gamma\Delta t\frac{g}{\theta^*}E(t^n, 2\gamma\Delta t)\left[\theta - \gamma\Delta t\frac{d\theta^*}{dz}w\right].$$

$\Longrightarrow$ Decoupling of discrete energy eq. from continuity and momentum eqs.

SL counterpart of BDF2 stage for VSE

$$\pi^{n+1}+\gamma_2\Delta t\,(c_p/c_v-1)\,\Pi^{n+2\gamma}\nabla\cdot\mathbf{u}^{n+1}=-\pi^*+(1-\gamma_3)[E\left(t^n,\Delta t\right)\Pi]+\gamma_3[E\left(t^n+2\gamma\Delta t,(1-2\gamma)\Delta t\right)\Pi],$$

$$u^{n+1}+\gamma_2\Delta t\;c_p\Theta^{n+2\gamma}\frac{\partial\pi}{\partial x}^{n+1}=(1-\gamma_3)[E\left(t^n,\Delta t\right)u]+\gamma_3[E\left(t^n+2\gamma\Delta t,(1-2\gamma)\Delta t\right)u],$$

$$w^{n+1}+\gamma_2\Delta t\left(c_p\Theta^{n+2\gamma}\frac{\partial\pi}{\partial z}^{n+1}-g\frac{\theta^{n+1}}{\theta^*}\right)=(1-\gamma_3)[E\left(t^n,\Delta t\right)w]+\gamma_3[E\left(t^n+2\gamma\Delta t,(1-2\gamma)\Delta t\right)w],$$

$$\theta^{n+1}+\gamma_2\Delta t\frac{d\theta^*}{dz}w^{n+1}=(1-\gamma_3)[E\left(t^n,\Delta t\right)\theta]+\gamma_3[E\left(t^n+2\gamma\Delta t,(1-2\gamma)\Delta t\right)\theta].$$

Inserting the discretized energy eq. into the discrete vertical momentum eq.:

$$\left(1+(\gamma_2\Delta t)^2\frac{g}{\theta^*}\frac{d\theta^*}{dz}\right)w^{n+1}+\gamma_2\Delta t\;c_p\Theta^{n+2\gamma}\frac{\partial\pi}{\partial z}^{n+1}=$$
$$(1-\gamma_3)[E\left(t^n,\Delta t\right)w]+\gamma_3[E\left(t^n+2\gamma\Delta t,(1-2\gamma)\Delta t\right)w]+$$
$$\gamma_2\Delta t\frac{g}{\theta^*}\left\{(1-\gamma_3)[E\left(t^n,\Delta t\right)\theta]+\gamma_3[E\left(t^n+2\gamma\Delta t,(1-2\gamma)\Delta t\right)\theta]\right\}.$$

$\Longrightarrow$ Decoupling of discrete energy eq. from continuity and momentum eqs.

Isomorphism btw. SWE and VSE SISL-TR step

$$h^{n+2\gamma} + \gamma\Delta t\ h^n\ \nabla\cdot\mathbf{u}^{n+2\gamma} = E\left(t^n, 2\gamma\Delta t\right)\left[h - \gamma\Delta t\ h\ \nabla\cdot\mathbf{u}\right],$$

$$\mathbf{u}^{n+2\gamma} + \gamma\Delta t\ g\nabla h^{n+2\gamma} = -\gamma\Delta t\ g\nabla b + E\left(t^n, 2\gamma\Delta t\right)\left\{\mathbf{u} - \gamma\Delta t\left[g(\nabla h + \nabla b)\right]\right\}.$$

$$\pi^{n+2\gamma} + \gamma\Delta t\ \left(c_p/c_v - 1\right)\Pi^n\nabla\cdot\mathbf{u}^{n+2\gamma} = -\pi^* + E\left(t^n, 2\gamma\Delta t\right)\left[\Pi - \gamma\Delta t\left(c_p/c_v - 1\right)\Pi\ \nabla\cdot\mathbf{u}\right],$$

$$u^{n+2\gamma} + \gamma\Delta t\ c_p\Theta^n\frac{\partial\pi}{\partial x}^{n+2\gamma} = E(t^n, 2\gamma\Delta t)\left[u - \gamma\Delta t\ c_p\Theta\frac{\partial\pi}{\partial x}\right],$$

$$\left(1 + (\gamma\Delta t)^2\frac{g}{\theta*}\frac{d\theta^*}{dz}\right)w^{n+2\gamma} + \gamma\Delta t c_p\Theta^n\frac{\partial\pi}{\partial z}^{n+2\gamma} = E(t^n, 2\gamma\Delta t)\left[w - \gamma\Delta t\left(c_p\Theta\frac{\partial\pi}{\partial z} - g\frac{\theta}{\theta*}\right)\right] + \gamma\Delta t\frac{g}{\theta*}E(t^n, 2\gamma\Delta t)\left[\theta - \gamma\Delta t\frac{d\theta^*}{dz}w\right].$$

h	$\longleftrightarrow$	$\pi,$
u	$\longleftrightarrow$	$u,$
v	$\longleftrightarrow$	$w.$

Isomorphism btw. SWE and VSE SISL-BDF2 step

$$h^{n+1} + \gamma_2 \Delta t\ h^{n+2\gamma}\ \nabla \cdot \mathbf{u}^{n+1} =$$
$$(1-\gamma_3) E(t^n, \Delta t) h$$
$$+\gamma_3 E(t^n + 2\gamma\Delta t, (1-2\gamma)\Delta t) h,$$

$$\pi^{n+1} + \gamma_2 \Delta t\,(c_p/c_v - 1)\,\Pi^{n+2\gamma}\,\nabla \cdot \mathbf{u}^{n+1} =$$
$$-\pi^* + (1-\gamma_3)[E\ (t^n, \Delta t)\ \Pi]$$
$$+\gamma_3[E\ (t^n + 2\gamma\Delta t, (1-2\gamma)\Delta t)\ \Pi],$$

$$u^{n+1} + \gamma_2 \Delta t\ c_p \Theta^{n+2\gamma} \frac{\partial \pi}{\partial x}^{n+1} =$$
$$(1-\gamma_3)[E\ (t^n, \Delta t)\ u]$$
$$+\gamma_3[E\ (t^n + 2\gamma\Delta t, (1-2\gamma)\Delta t)\ u],$$

$$\mathbf{u}^{n+1} + \gamma_2 \Delta t\ g \nabla h^{n+1} =$$
$$-\gamma_2 \Delta t\ g \nabla b$$
$$+(1-\gamma_3) E(t^n, \Delta t)\mathbf{u}$$
$$+\gamma_3 E(t^n + 2\gamma\Delta t, (1-2\gamma)\Delta t)\mathbf{u}.$$

$$\left(1 + (\gamma_2\Delta t)^2 \frac{g}{\theta *} \frac{d\theta^*}{dz}\right) w^{n+1} + \gamma_2 \Delta t\ c_p \Theta^{n+2\gamma} \frac{\partial \pi}{\partial z}^{n+1} =$$
$$(1-\gamma_3)[E\ (t^n, \Delta t)\ w] + \gamma_3[E\ (t^n + 2\gamma\Delta t, (1-2\gamma)\Delta t)\ w] +$$
$$\gamma_2 \Delta t \frac{g}{\theta *} \{(1-\gamma_3)[E\ (t^n, \Delta t)\ \theta] + \gamma_3[E\ (t^n + 2\gamma\Delta t, (1-2\gamma)\Delta t)\ \theta]\}$$

$$h \longleftrightarrow \pi,$$
$$u \longleftrightarrow u,$$
$$v \longleftrightarrow w.$$

DG space discretization

- Defined a tassellation $\mathcal{T}_h = \{K_l\}_{l=1}^{N}$ of domain Ω and chosen $\forall K_l \in \mathcal{T}_h$ two integers $p_l^h \geq 0$, $p_l^u \geq 0$, at each time level t^n, we are looking for approximate solution s.t.

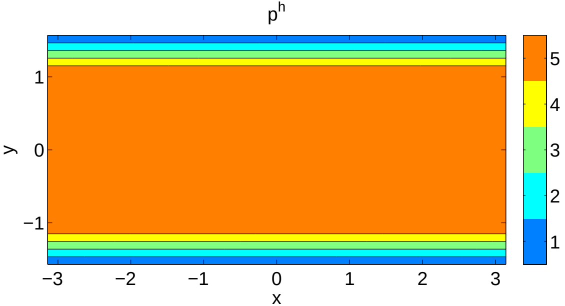
$$\begin{aligned} h^n, \pi^n, \theta^n &\in H_h := \left\{ f \in L^2(\Omega) \; : \; f|_{K_l} \in \mathbb{Q}_{p_l^h}(K_l) \right\} \\ \boldsymbol{u}^n &\in V_h := \left\{ g \in L^2(\Omega) \; : \; g|_{K_l} \in \mathbb{Q}_{p_l^u}(K_l) \right\}^2 , \end{aligned}$$

- modal bases are used to span H_h, V_h;
- L^2 projection against test functions (chosen equal to the basis functions),
- introduction of (centered) numerical fluxes,
- substitution of velocity d.o.f. from momentum eqs. into the continuity eq.,
- give raise, at each SI step, to a discrete (vector) Helmholtz equation in the fluid depth / pressure unknown only,

  i.e. a sparse block structured nonsymmetric linear system is solved by GMRES with *block* diagonal (for the moment) preconditioning.

Static p-adaptation: reduced *p* as counterpart of reduced grid

p_l can be imposed locally in order to control the local Courant number:



$\Longrightarrow$ this leads to significant efficiency improvement:

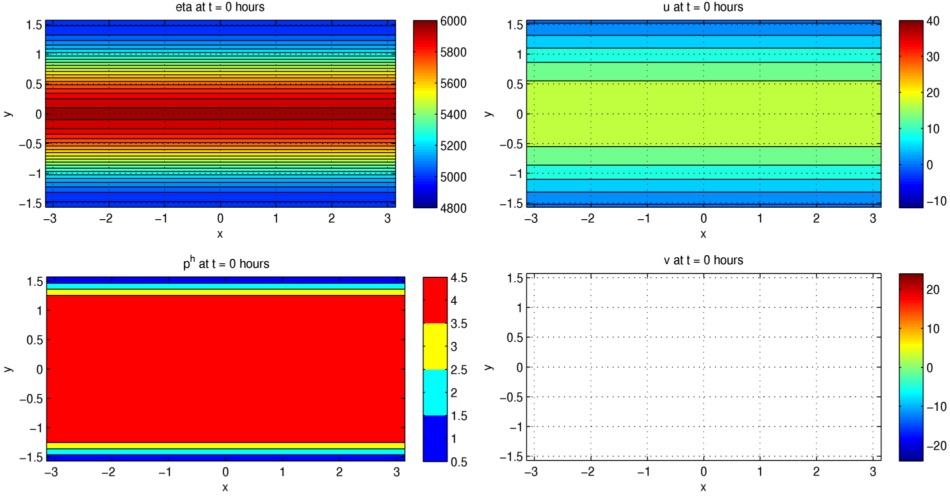
$$\frac{\#\text{gmres-iterations}(p^h = \text{adapted})}{\#\text{gmres-iterations}(p^h = \text{uniform})} \approx 10\% \div 50\%,$$

Numerical Validation

Combination of static + dynamic p-adaptation: Williamson's test 5

64 × 32 elements, $\max p^h = 4$, $\Delta t = 900s$ ($C_{cel} \approx 83$ without adaptivity)

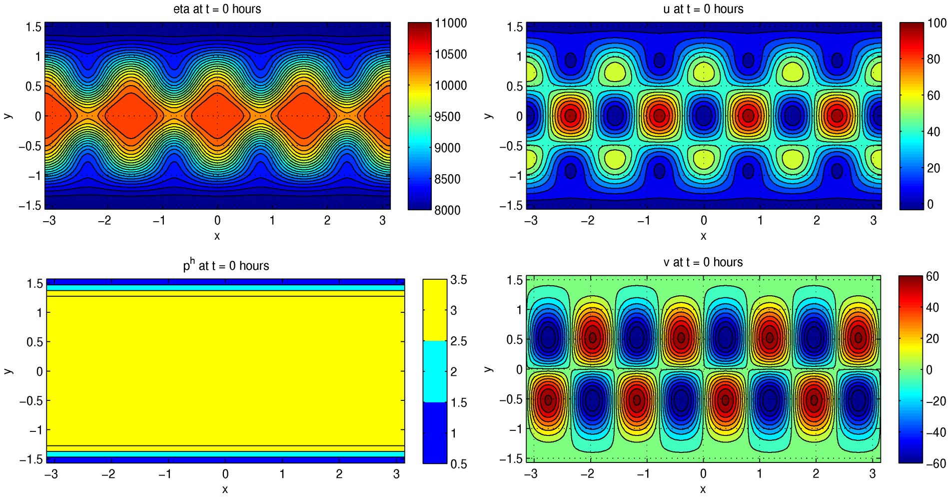
$$\frac{\#\text{gmres-iterations}(p^h = \text{adapted})}{\#\text{gmres-iterations}(p^h = \text{uniform})} \approx 13\%, \qquad \Delta^n_{dof} = \frac{\sum_{l=1}^{N}(p^n_l + 1)^2}{N(p_{max}+1)^2} \approx 45\%$$



Combination of static + dynamic p-adaptation: Williamson's test 6

64×32 elements, $\quad \max p^h = 4, \quad \Delta t = 900s \quad (C_{cel} \approx 83$ without adaptivity)

$$\frac{\#\text{gmres-iterations}(p^h = \text{adapted})}{\#\text{gmres-iterations}(p^h = \text{uniform})} \approx 13\%, \qquad \Delta^n_{dof} = \frac{\sum_{l=1}^{N}(p_l^n + 1)^2}{N(p_{max}+1)^2} \approx 45\%$$



Williamson's test 6: time convergence rate

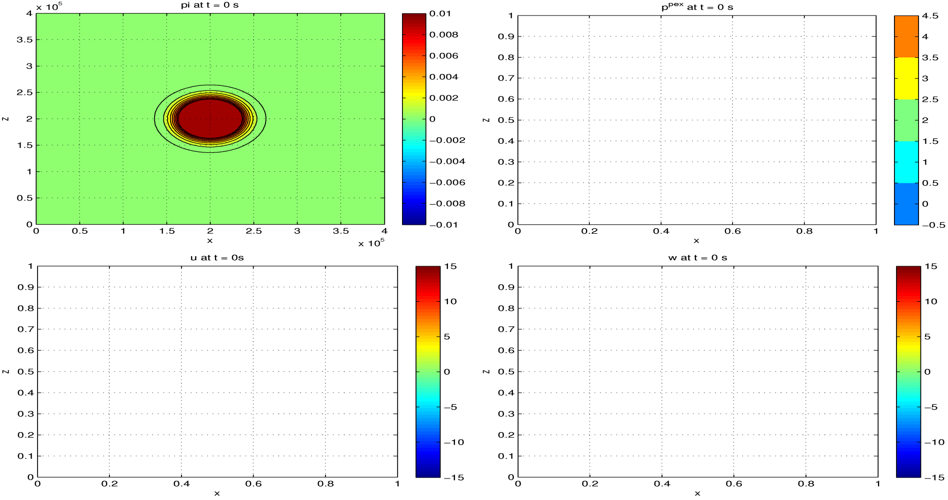
Relative errors for different number of elements, with respect to NCAR spectral model solution at resolution T511:

$N_x \times N_y$	Δt[min]	$l_1(h)$	$l_2(h)$	$l_\infty(h)$
10×5	60	2.92×10^{-2}	3.82×10^{-2}	6.75×10^{-2}
20×10	30	5.50×10^{-3}	6.80×10^{-3}	1.11×10^{-2}
40×20	15	1.40×10^{-3}	1.80×10^{-3}	3.20×10^{-3}

$N_x \times N_y$	Δt[min]	$l_1(u)$	$l_2(u)$	$l_\infty(u)$
10×5	60	4.065×10^{-1}	3.775×10^{-1}	2.305×10^{-1}
20×10	30	7.79×10^{-2}	7.33×10^{-2}	5.67×10^{-2}
40×20	15	2.04×10^{-2}	1.95×10^{-2}	1.76×10^{-2}

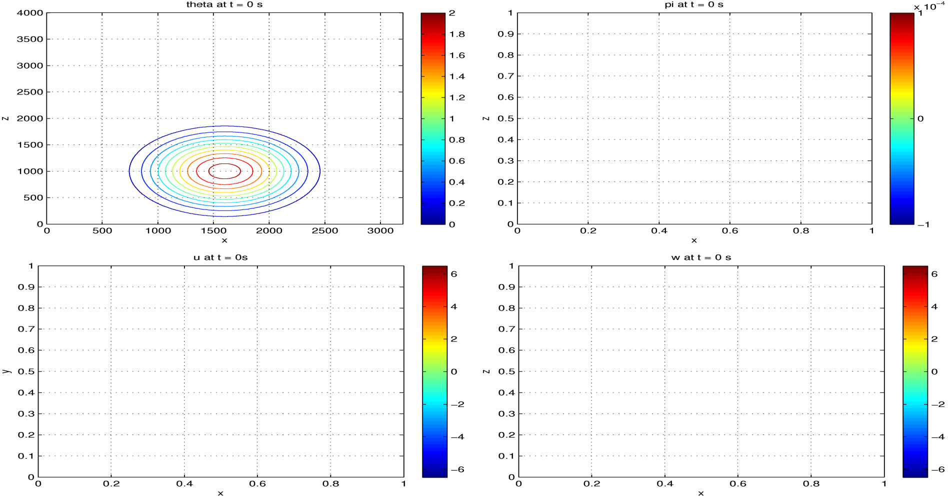
p-adaptive acoustic wave propagation

$$50 \times 50 \text{ elements}, \;\; p^{\pi} = 4, \;\; p^{u} = 5, \quad \Delta t = 10 \text{ s}, \;\; C_{vel} \approx 2$$



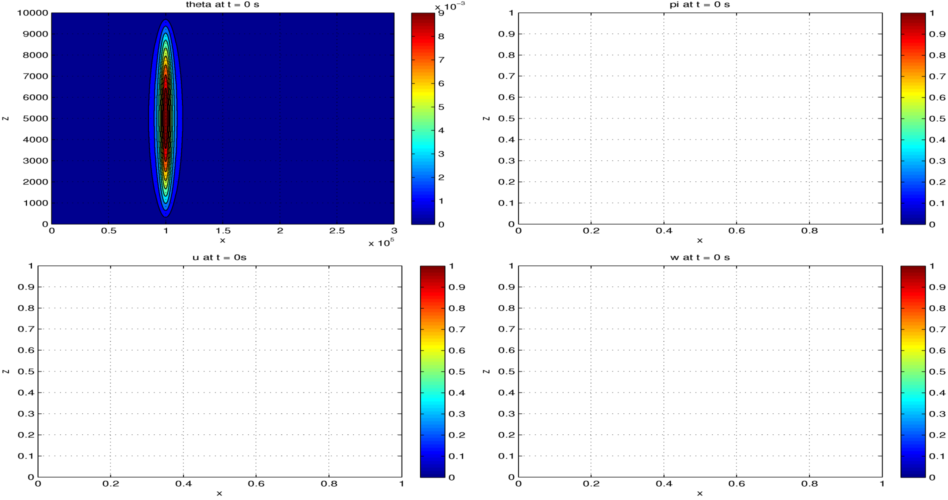
Warm bubble test (Carpenter et al., MWR 1990)

64×80 elements, $p^{\pi} = 4$, $p^{u} = 5$, $\Delta t = 0.5$ s, $C \approx 17$



Inertia-gravity wave (Skamarock and Klemp, MWR 1994)

300×10 elements, $p^{\pi} = 4$, $p^{u} = 5$, $\Delta t = 15$ s, $C \approx 25$



Conclusions, open issues and future perspectives

- In summary:
  - a novel TRBDF2-based SISL discretization has been presented within the DG framework for the rotating SWE as well as for the Euler equations on a vertical slice, that can be effectively applied to all geophysical scale flows.
  - the resulting scheme is
    - unconditionally stable,
    - full second order accurate in time,
    - arbitrary high order in space,
    - adapting the number of degrees of freedom in each element in order to balance accuracy and computational cost,
    - even if presented on structured meshes, extendeble to arbitrary non-structured;
  - numerical experiments prove the effectiveness of the proposed scheme.
- Now on the way:
  - introduction of the topography;
  - completion of implementation of p-adaptivity on the compressible equations;
  - improvement of the linear solver for the SI step: preconditioning strategy, from block diagonal to ILU.
- Future perspectives:
  - comparison with other stiff time integration techniques (e.g. Rosenbrock and exponential integrators);
  - parallelization strategy;
  - integration of SISLDG discretizations of SWE and VSE to develop the 3D nonhydrostatic dynamical core for RegCM;

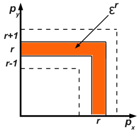
Dynamic p-adaptation strategy

- p-adaptivity easier by the use of modal bases: here tensor products of Legendre polynomials;
- hence, the representation for a model variable α becomes ($l = (l_x, l_y)$ multi-index):

$$\alpha(\boldsymbol{x})\big|_{K_l} = \sum_{k=1}^{p_l^\alpha+1} \sum_{l=1}^{p_l^\alpha+1} \alpha_{l,k,l}\psi_{l_x,k}(x)\psi_{l_y,l}(y).$$

- and its 2-norm is given by (in planar geometry):

$$\mathcal{E}_l^{tot} = \sum_{k,l=1}^{p_l^\alpha+1} \alpha_{l,k,l}^2 = \sum_{r=1}^{p_l^\alpha+1} \mathcal{E}_l^r, \quad \mathcal{E}_l^r := \sum_{max(k,l)=r} \alpha_{l,k,l}^2,$$



- while the quantity $w_l^r = \sqrt{\frac{\mathcal{E}_l^r}{\mathcal{E}_l^{tot}}}$ will measure the relative 'weight' of the $r-$ degree modes
- *Given an error tolerance $\epsilon_l > 0$ for all $l = 1, \ldots, N$, at each time step repeat following steps:*

  1) *compute w_{p_i}*

  2.1) *if $w_{p_i} \geq \epsilon_i$, then*
  - 2.1.1) *set $p_i(\alpha) := p_i(\alpha) + 1$*
  - 2.1.2) *set $\alpha_{i,p_i} = 0$, exit the loop and go the next element*

  2.2) *if instead $w_{p_i} < \epsilon_i$, then*
  - 2.2.1) *compute w_{p_i-1}*
  - 2.2.2) *if $w_{p_i-1} \geq \epsilon_i$, exit the loop and go the next element*
  - 2.2.3) *else if $w_{p_i-1} < \epsilon_i$, set $p_i(\alpha) := p_i(\alpha) - 1$ and go back to 2.2.1.*

Läuter unsteady flow: time convergence rate

Errors after 5 days, $p^h = 4$, $p^u = 5$, $4 < C_{cel} < 26$, $1.25 < C_{vel} < 8$.

$N_x \times N_y$	$\Delta t\ [s]$	$l_1(h)$	$l_2(h)$	$l_\infty(h)$	q_2^{emp}
10×5	3600	5.456×10^{-3}	6.120×10^{-3}	9.537×10^{-3}	-
20×10	1800	1.246×10^{-3}	1.397×10^{-3}	2.143×10^{-3}	2.1
40×20	900	3.039×10^{-4}	3.410×10^{-4}	5.207×10^{-4}	2.0
80×40	450	7.548×10^{-5}	8.475×10^{-5}	1.292×10^{-4}	2.0

$N_x \times N_y$	$\Delta t\ [s]$	$l_1(u)$	$l_2(u)$	$l_\infty(u)$	q_2^{emp}
10×5	3600	6.567×10^{-2}	7.848×10^{-2}	1.670×10^{-1}	-
20×10	1800	1.665×10^{-2}	1.994×10^{-2}	3.931×10^{-2}	2.0
40×20	900	4.210×10^{-3}	5.032×10^{-3}	9.811×10^{-3}	2.0
80×40	450	1.057×10^{-3}	1.261×10^{-3}	2.452×10^{-3}	2.0

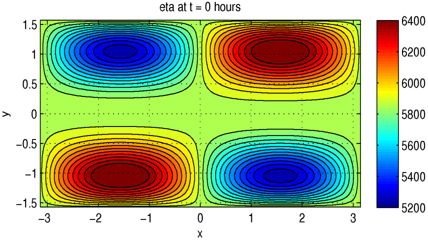
$N_x \times N_y$	$\Delta t\ [s]$	$l_1(v)$	$l_2(v)$	$l_\infty(v)$	q_2^{emp}
10×5	3600	1.174×10^{-1}	1.198×10^{-1}	2.316×10^{-1}	-
20×10	1800	2.939×10^{-2}	3.002×10^{-2}	5.561×10^{-2}	2.0
40×20	900	7.336×10^{-3}	7.497×10^{-3}	1.390×10^{-2}	2.0
80×40	450	1.833×10^{-3}	1.874×10^{-3}	3.464×10^{-3}	2.0

Cross-polar flow (McDonald's and Bates, MWR 1989)

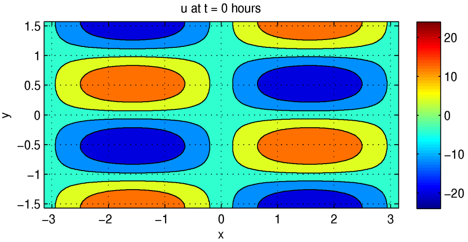
50×25 elements, $p^h = 4, p^u = 5$ $\Delta t = 900s$ ($C_{cel} \approx 43$, $C_{vel} \approx 4$ close to poles without adaptivity).

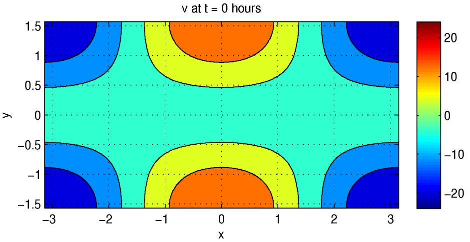
[days]	$l_1(h)$	$l_2(h)$	$l_\infty(h)$
10	5.848×10^{-6}	1.338×10^{-5}	1.101×10^{-4}
15	8.365×10^{-6}	1.871×10^{-5}	1.014×10^{-4}

Table: Relative errors for h between statically adaptive and uniform solution.



$$\frac{\#\text{gmres-iterations}(p^h = \text{adapted})}{\#\text{gmres-iterations}(p^h = \text{uniform})} \approx 43\%$$





Williamson's test 2: space convergence rate

Relative errors at time $t_f = 10$ days, different pol. deg., 10×5 elements, $\alpha = \pi/2 - 0.05$.

p^h	p^u	Δt [s]	$l_1(h)$	$l_2(h)$	$l_\infty(h)$
2	3	4800	5.558×10^{-3}	6.805×10^{-3}	1.914×10^{-2}
3	4	3600	6.017×10^{-4}	8.176×10^{-4}	2.569×10^{-3}
4	5	2880	1.743×10^{-5}	2.405×10^{-5}	9.024×10^{-5}
5	6	2400	1.586×10^{-6}	2.281×10^{-6}	1.058×10^{-5}
6	7	2057	8.829×10^{-8}	1.206×10^{-7}	4.926×10^{-7}
7	8	1800	1.246×10^{-8}	1.590×10^{-8}	4.158×10^{-8}
8	9	1600	5.641×10^{-9}	5.952×10^{-9}	6.320×10^{-9}

p^h	p^u	Δt [s]	$l_1(u)$	$l_2(u)$	$l_\infty(u)$
2	3	4800	6.351×10^{-2}	6.432×10^{-2}	1.143×10^{-1}
3	4	3600	9.505×10^{-3}	1.037×10^{-2}	2.106×10^{-2}
4	5	2880	4.288×10^{-4}	4.887×10^{-4}	2.393×10^{-3}
5	6	2400	4.598×10^{-5}	4.830×10^{-5}	1.706×10^{-4}
6	7	2057	2.057×10^{-6}	2.262×10^{-6}	5.879×10^{-6}
7	8	1800	2.162×10^{-7}	2.358×10^{-7}	6.428×10^{-7}
8	9	1600	2.013×10^{-8}	2.276×10^{-8}	3.268×10^{-8}

Williamson's test 2: space convergence rate

Relative errors at time $t_f = 10$ days, different pol. deg., 10×5 elements, $\alpha = \pi/2 - 0.05$.

p^h	p^u	Δt [s]	$l_1(v)$	$l_2(v)$	$l_\infty(v)$
2	3	4800	1.001×10^{-1}	1.016×10^{-1}	2.698×10^{-1}
3	4	3600	1.859×10^{-2}	1.823×10^{-2}	6.848×10^{-2}
4	5	2880	7.376×10^{-4}	7.428×10^{-4}	2.884×10^{-3}
5	6	2400	8.185×10^{-5}	8.307×10^{-5}	2.574×10^{-4}
6	7	2057	3.074×10^{-6}	3.173×10^{-6}	1.123×10^{-5}
7	8	1800	3.370×10^{-7}	3.432×10^{-7}	1.323×10^{-6}
8	9	1600	2.175×10^{-8}	2.317×10^{-8}	5.124×10^{-8}

Cold bubble test (Straka et al., IJNMF 1993)

80×20 elements, $p^\pi = 3$, $p^u = 3$, $\Delta t = 0.5$ s