Arbitrary-Order Hybrid Finite-Element Methods for Geophysical Flows

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Tempest Framework

An Earth-system modeling framework formulated using an explicitly evaluated horizontal discretization and implicitly evaluated vertical discretization (HEVI).

Built using block-based refinement analogous to Colella and Berger (1989). Geometry is included via Riemannian metric terms.

Allows for effectively arbitrary (and nested) quadrilateral structures in the horizontal, with current support for Cartesian and Cubed-Sphere grids.

Numerical methods formulated independent of grid.

Figure: A cubed-sphere refinement patch over California.

HARDCore Dynamics

High-order Adaptively Refined Dynamical Core (HARDCore).

Currently supports **Spectral Element (SE)**, **Discontinuous Galerkin (DG)** and **Flux Reconstruction (FR)** type methods with arbitrary order-of-accuracy in horizontal and vertical. Goal is to also support upwind and central **Finite Volume (FV)** methods.

Non-hydrostatic dynamics with the full set of fluid equations, with additional support for a spatially-variable reference profile.

Figure: A cubed-sphere refinement patch over California.

Two New Technologies

Part 1

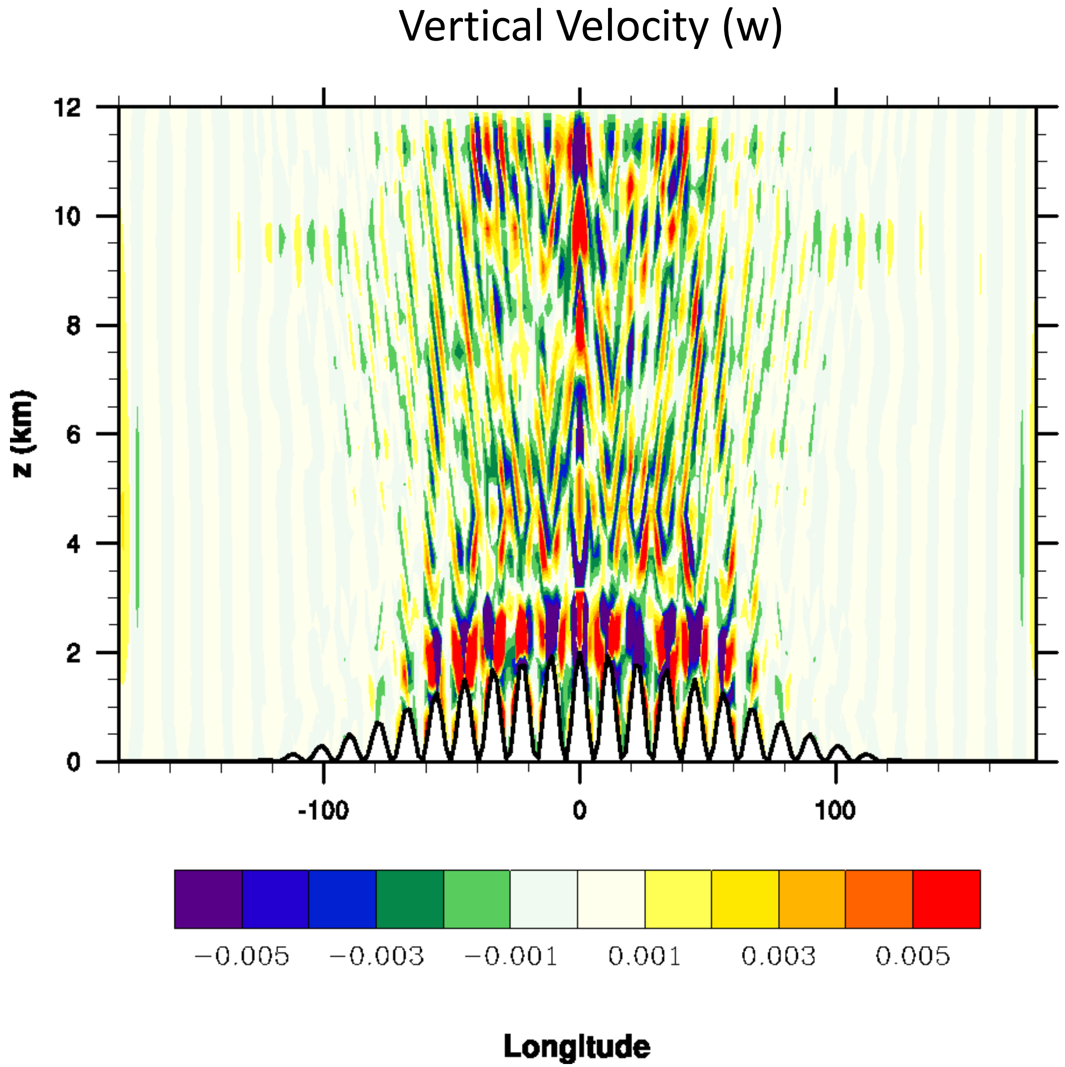
Non-conservative discontinuous Galerkin

Part 2

A new approach to a high-order vertical discretization

Motivation: Spurious Vertical Velocity

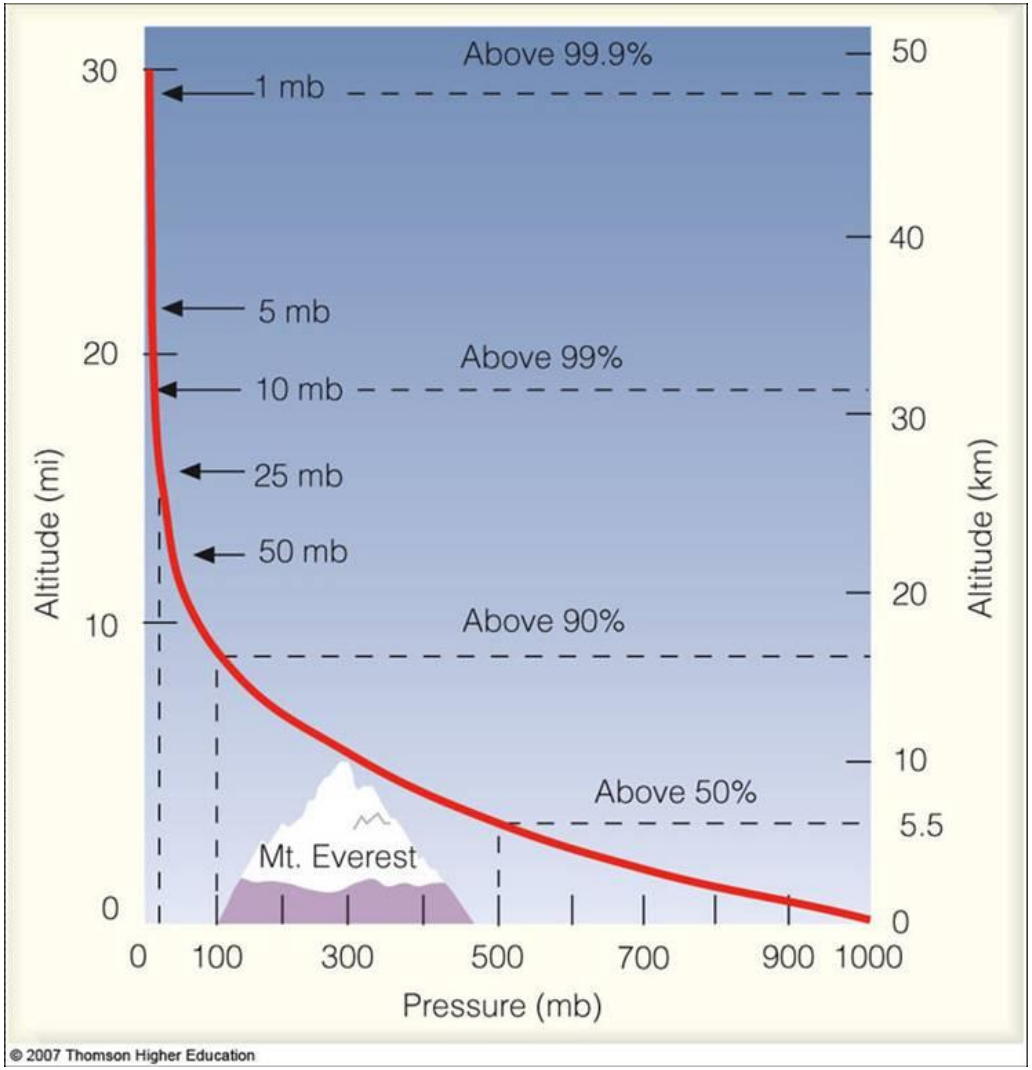
Figure: An initially steady atmosphere (no initial velocities) with sharply varying bottom topography leads to spurious vertical velocities because of inaccuracy in the evaluation of the horizontal pressure gradient term. Here the vertical velocity errors are on the order of 0.5 cm/s.



Motivation: Vertical Stratification

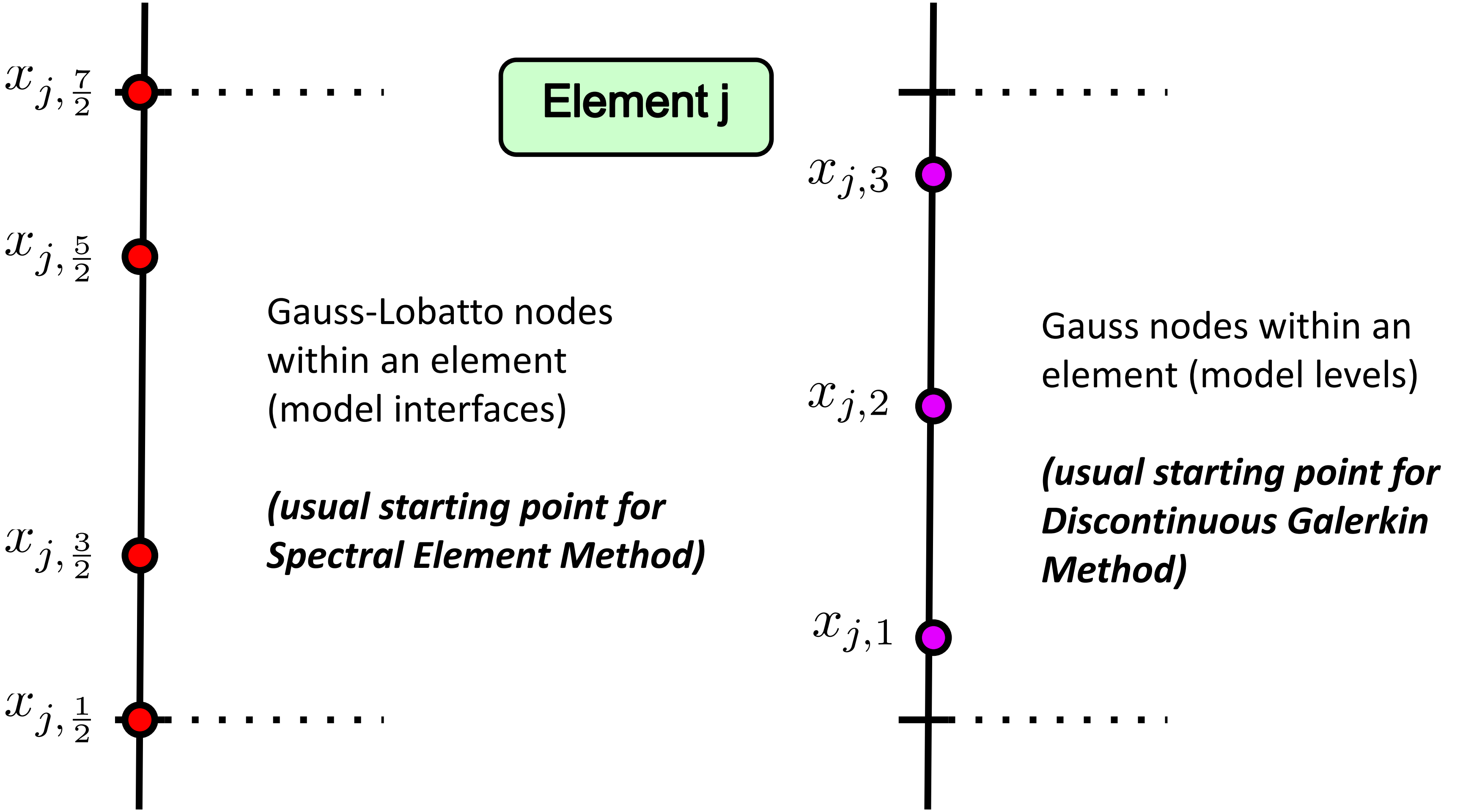
Figure: A low-order vertical coordinate does a poor job of resolving hydrostatic balance due to approximately exponential decay of the pressure and density terms.

Most non-hydrostatic numerical models instead use a “reference profile” – a spatially variable steady background state – which is subtracted from the equations of motion.



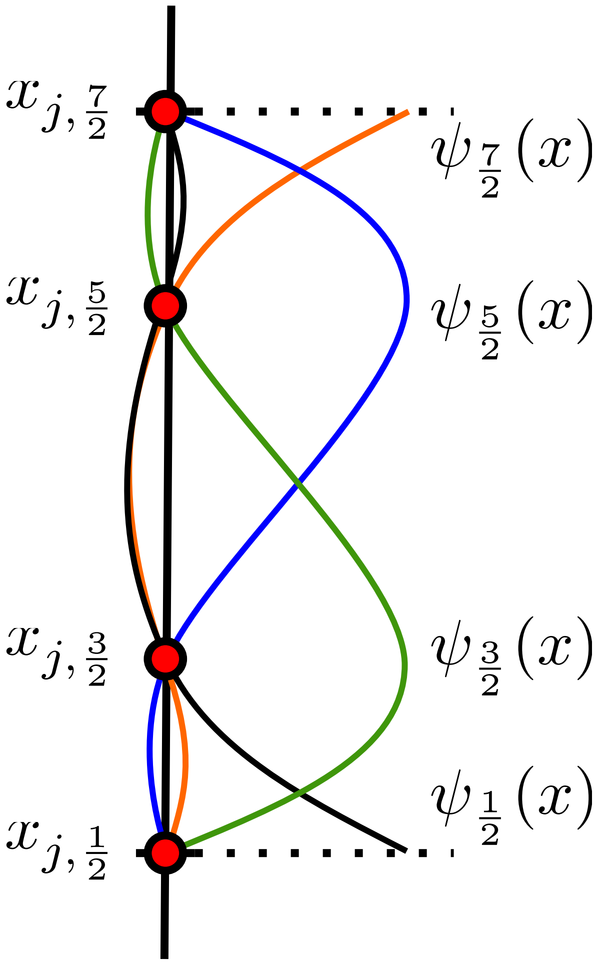
Nodal Finite Element Method

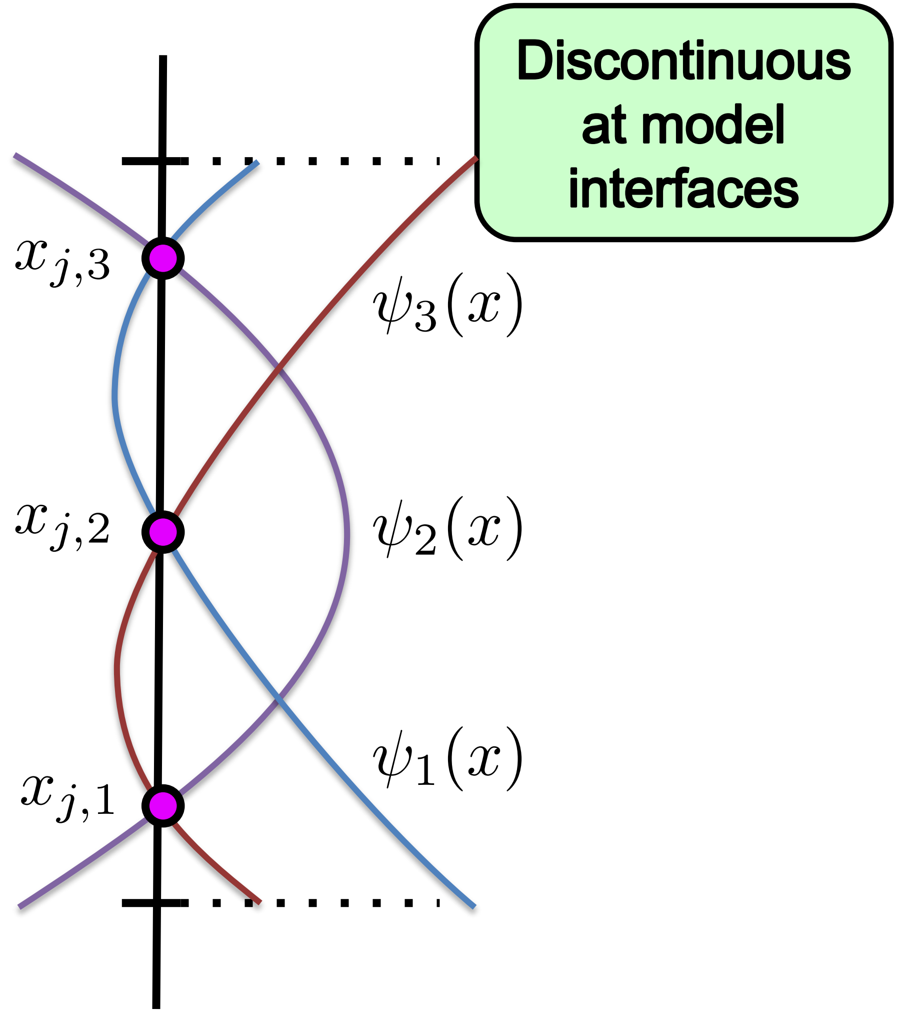
Finite-element methods are a common framework for high-order methods. A typical continuous basis using the spectral element method is situated on Gauss-Lobatto nodes. A discontinuous basis is defined on Gauss nodes.



Nodal Finite Element Method

Each node is then associated with a unique basis function which is 1 at a particular node and 0 at all other nodes.





Finite Element Method

1D Conservation Eq'n

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x} F(q) = 0$$

Examining the 1D conservation equation, one can obtain a ***differential formulation*** of the finite element method on nodes.

Step 1: Calculate fluxes F_j at each nodal location x_j.

Step 2: Fit an interpolating polynomial through nodal fluxes.

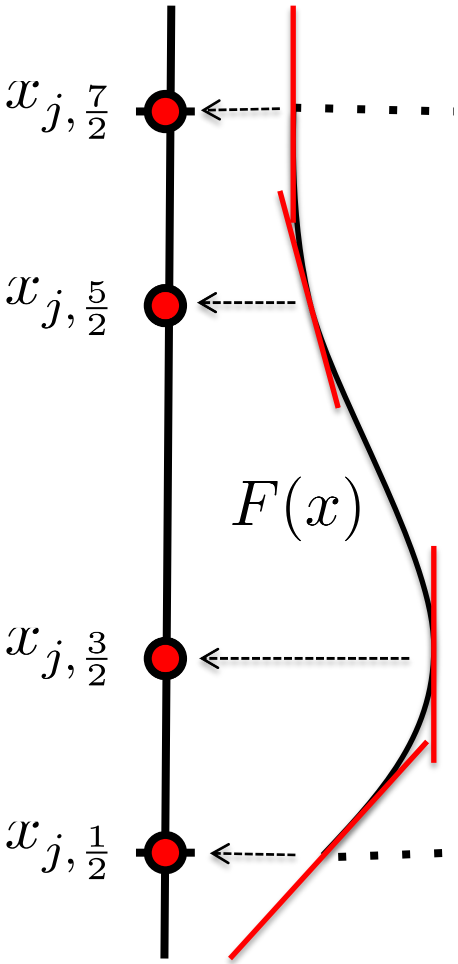
Step 3: Differentiate the interpolating polynomial at nodes.

On a per element level, this approach is identical to the finite element formulation with free-flux (unspecified) boundary conditions.

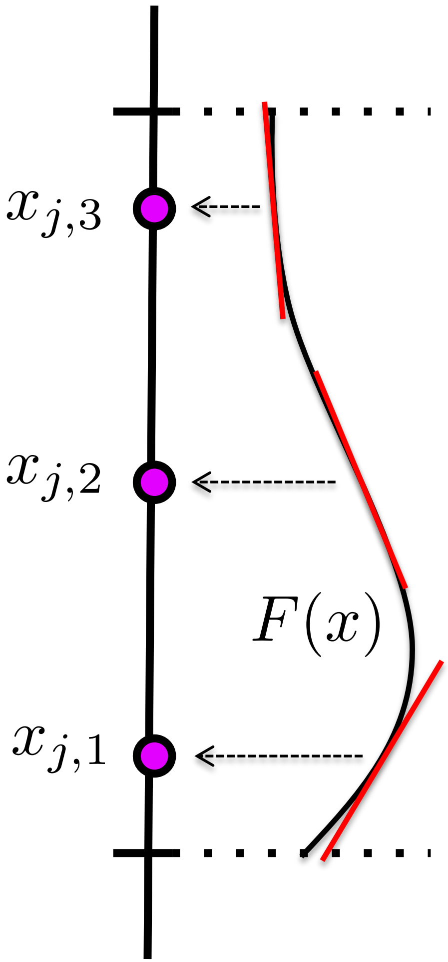
To impose coupling between elements, we require Direct Stiffness Summation (DSS) for the spectral element method or **???** for the Discontinuous Galerkin method.

Finite Element Method

The derivatives of the flux interpolant are used for updating the pointwise values of the state. However, there must be a mechanism for coupling elements.



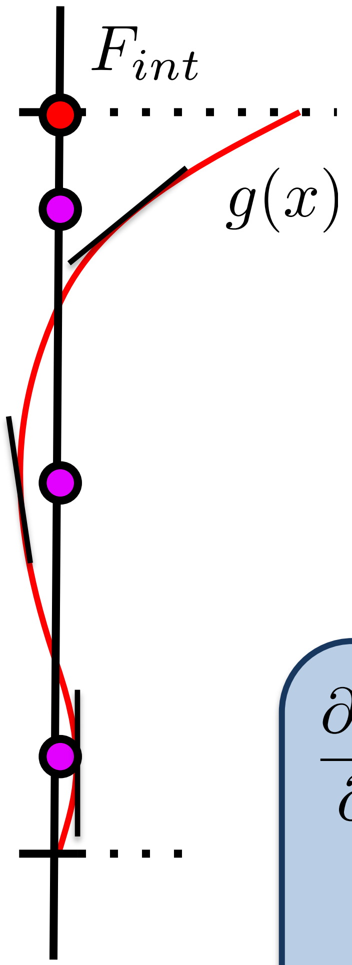
DSS: Average derivatives on shared boundary nodes. Natural method for coupling elements.



Unclear how elements should be coupled in differential form

– hence why DG is typically formulated for a conservation law, where coupling is via Riemann solver

Flux Reconstruction Methods (Huynh 2007)



The interface flux polynomial is 1 at one element interface, 0 at the other interface and approximates 0 throughout the element.

Central flux:

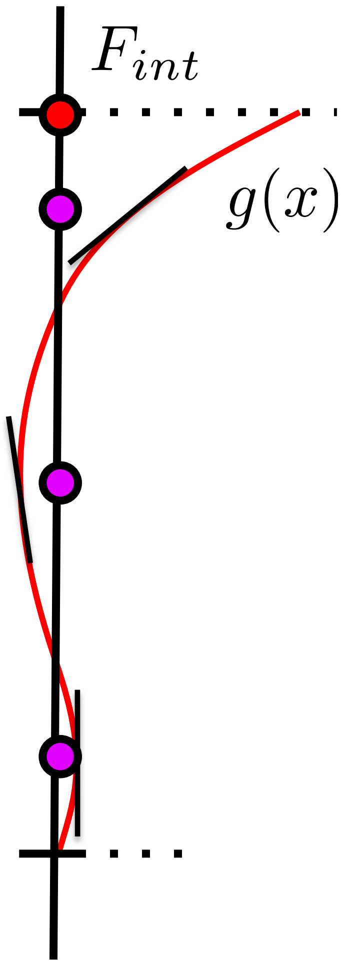
$$F_{int} = \frac{F_L + F_R}{2}$$

Local Lax-Friedrichs flux:

$$F_{int} = \frac{F_L + F_R}{2} - \frac{|\lambda|}{2}(q_R - q_L)$$

$$\frac{\partial q_j}{\partial t} = - \left.\frac{\partial F}{\partial x}\right|_{x_j} + \left(F_{int,top} - F(x_{top})\right) \left.\frac{\partial g_{top}}{\partial x}\right|_{x_j} + \left(F_{int,bot} - F(x_{bot})\right) \left.\frac{\partial g_{bot}}{\partial x}\right|_{x_j}$$

Flux Reconstruction Methods (Huynh 2007)



The flux reconstruction approach motivates the notion of a **discrete derivative** which is consistent with the discontinuous Galerkin formulation.

$$\frac{\partial q}{\partial x} \approx \left.\frac{\partial \hat{q}}{\partial x}\right|_{x_j} + (q_{int,top} - \hat{q}(x_{top})) \left.\frac{\partial g_{top}}{\partial x}\right|_{x_j} + (q_{int,bot} - \hat{q}(x_{bot})) \left.\frac{\partial g_{bot}}{\partial x}\right|_{x_j}$$

Here $\hat{q}$ denotes the interpolating polynomial through interior nodes. The value at the top and bottom of the element can be obtained via a central approximation:

$$q_{int,top} = \frac{\hat{q}_j(x_{top}) + \hat{q}_{j+1}(x_{bot})}{2}$$

Non-Conservative DG

The notion of a "DG derivative operator" from the flux reconstruction method allows DG to be applied to the non-conservative equations of motion. Since the density evolution equation is conservative, the flux reconstruction approach reduces to DG in this case.

$$\frac{\partial \rho}{\partial t} + \nabla_h \cdot (\rho \mathbf{u}_h) = -\frac{\partial}{\partial \xi}(\rho u^\xi)$$

$$\frac{\partial \mathbf{u}_h}{\partial t} + \mathbf{u}_h \cdot \nabla_h \mathbf{u}_h + u^\xi \frac{\partial \mathbf{u}_h}{\partial \xi} + \frac{1}{\rho}\nabla_h p + f\mathbf{k} \times \mathbf{u}_h = 0$$

$$\frac{\partial u^\xi}{\partial t} + \mathbf{u}_h \cdot \nabla_h u^\xi = -\frac{\theta}{a^2}\frac{\partial \Pi}{\partial \xi} - \frac{g}{a}$$

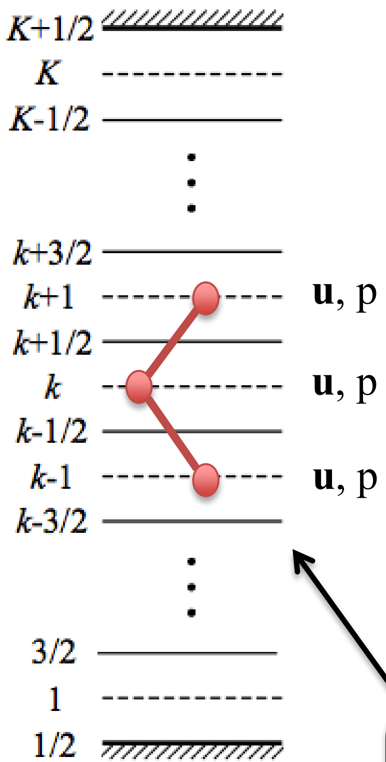
$$\frac{\partial \theta}{\partial t} + \mathbf{u}_h \cdot \nabla_h \theta = -u^\xi \frac{\partial \theta}{\partial \xi}$$

Explicit Timestepping

Implicit Timestepping

The Need for Staggering

Unstaggered grid (Arakawa A-grid)



Linearized vertical velocity equation:

$$\frac{\partial w}{\partial t} + \frac{g}{\overline{\rho}}\rho + \frac{1}{\overline{\rho}}\frac{\partial p}{\partial z} = 0$$

Density perturbation (affects magnitude)

Ignore for now

Pressure perturbation (affects propagation)

Discretization on the unstaggered grid:

$$\frac{\partial w}{\partial t} + \frac{1}{\overline{\rho}}\left(\frac{p_{k+1} - p_{k-1}}{z_{k+1} - z_{k-1}}\right) = 0$$

Supports 2Δx mode!

Result holds for ALL A-Grid Numerical Methods, including arbitrary order FEM!

Vertical Staggering

Linearized vertical velocity equation:

$$\frac{\partial w}{\partial t} + \frac{g}{\overline{\rho}}\rho + \frac{1}{\overline{\rho}}\frac{\partial p}{\partial z} = 0$$

Discretization on staggered grid:

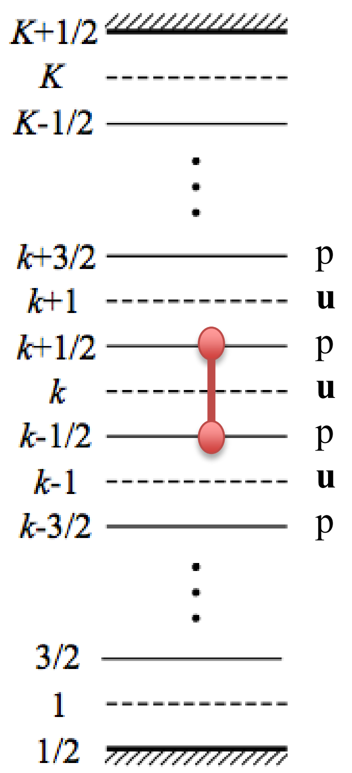
$$\frac{\partial w_k}{\partial t} + \frac{1}{\overline{\rho}}\left(\frac{p_{k+1/2} - p_{k-1/2}}{z_{k+1/2} - z_{k-1/2}}\right) = 0$$

No separation of odd/even modes in w/p. Still can get computational mode in other variables depending on staggering.

Also see:

- Tokioka (1978)
- Arakawa and Moorthi (1988)
- Arakawa and Konor (1996)

Staggered grid (Like an Arakawa C-grid)



Vertical Staggering

There are many possible choices of vertical staggering besides the one shown above:

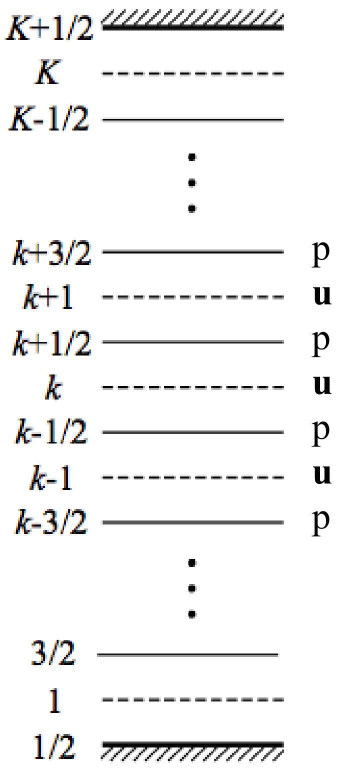
- For non-hydrostatic system there are **5 prognostic variables**.
- We can choose **any two thermodynamic** variables from ρ, p, T, θ, etc.

Some have computational modes. Others do not.

- Accurate representation of waves (acoustic, inertia-gravity, Rossby) only exhibited by certain choices of staggering and prognostic variables.

- Analyzed by **Thuburn and Woollings (2005)** in three coordinate systems.

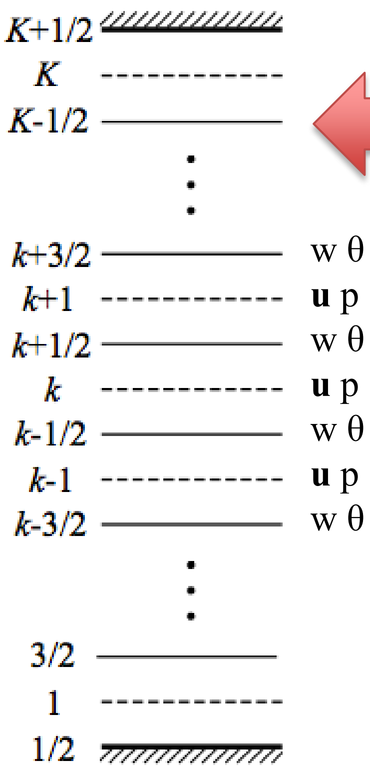
Staggered grid
(Like an Arakawa C-grid)



Vertical Staggering

Thuburn and Woollings (2005), Thuburn (2006) and Toy and Randall (2006) showed that there are few choices of vertical staggering which have optimal wave-propagation properties. In z-coordinates, there are two options:

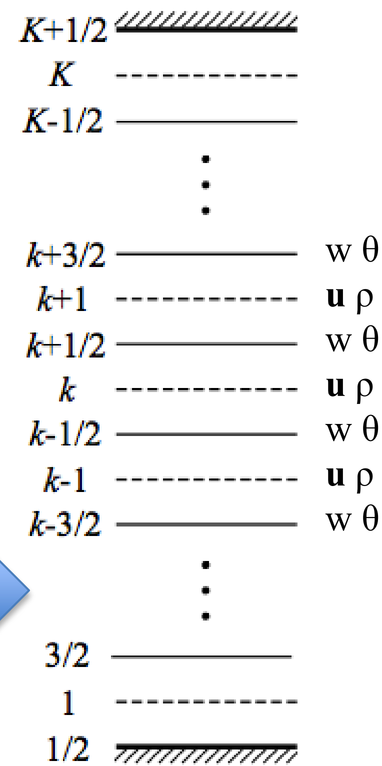
(wθ,uvp) grid



Vertical velocity and potential temperature at model interfaces, velocity and pressure at model levels. **Conservation of mass very difficult under this formulation!**

Vertical velocity and potential temperature at model interfaces, velocity and ***density*** at model levels. Exner pressure used for buoyancy term. **Conservation of mass easy under this formulation!**

$(w\theta,uv\rho)^{\Pi}$ grid



Non-Conservative DG

By using the differential form of the FEM in conjunction with the Charney-Phillips staggering then yields a **generalized vertical discretization which supports arbitrary order-of-accuracy**.

$$\frac{\partial \rho}{\partial t} + \nabla_h \cdot (\rho \mathbf{u}_h) = -\frac{\partial}{\partial \xi}(\rho u^\xi)$$

$$\frac{\partial \mathbf{u}_h}{\partial t} + \mathbf{u}_h \cdot \nabla_h \mathbf{u}_h + u^\xi \frac{\partial \mathbf{u}_h}{\partial \xi} + \frac{1}{\rho}\nabla_h p + f\mathbf{k} \times \mathbf{u}_h = 0$$

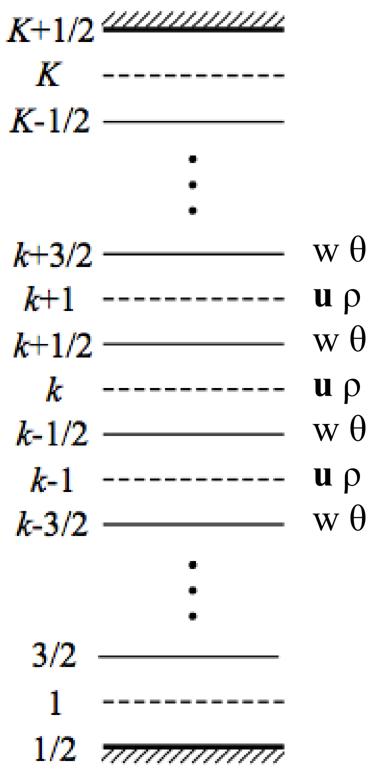
$$\frac{\partial u^\xi}{\partial t} + \mathbf{u}_h \cdot \nabla_h u^\xi = -\frac{\theta}{a^2}\frac{\partial \Pi}{\partial \xi} - \frac{g}{a}$$

$$\frac{\partial \theta}{\partial t} + \mathbf{u}_h \cdot \nabla_h \theta = -u^\xi \frac{\partial \theta}{\partial \xi}$$

Explicit Timestepping

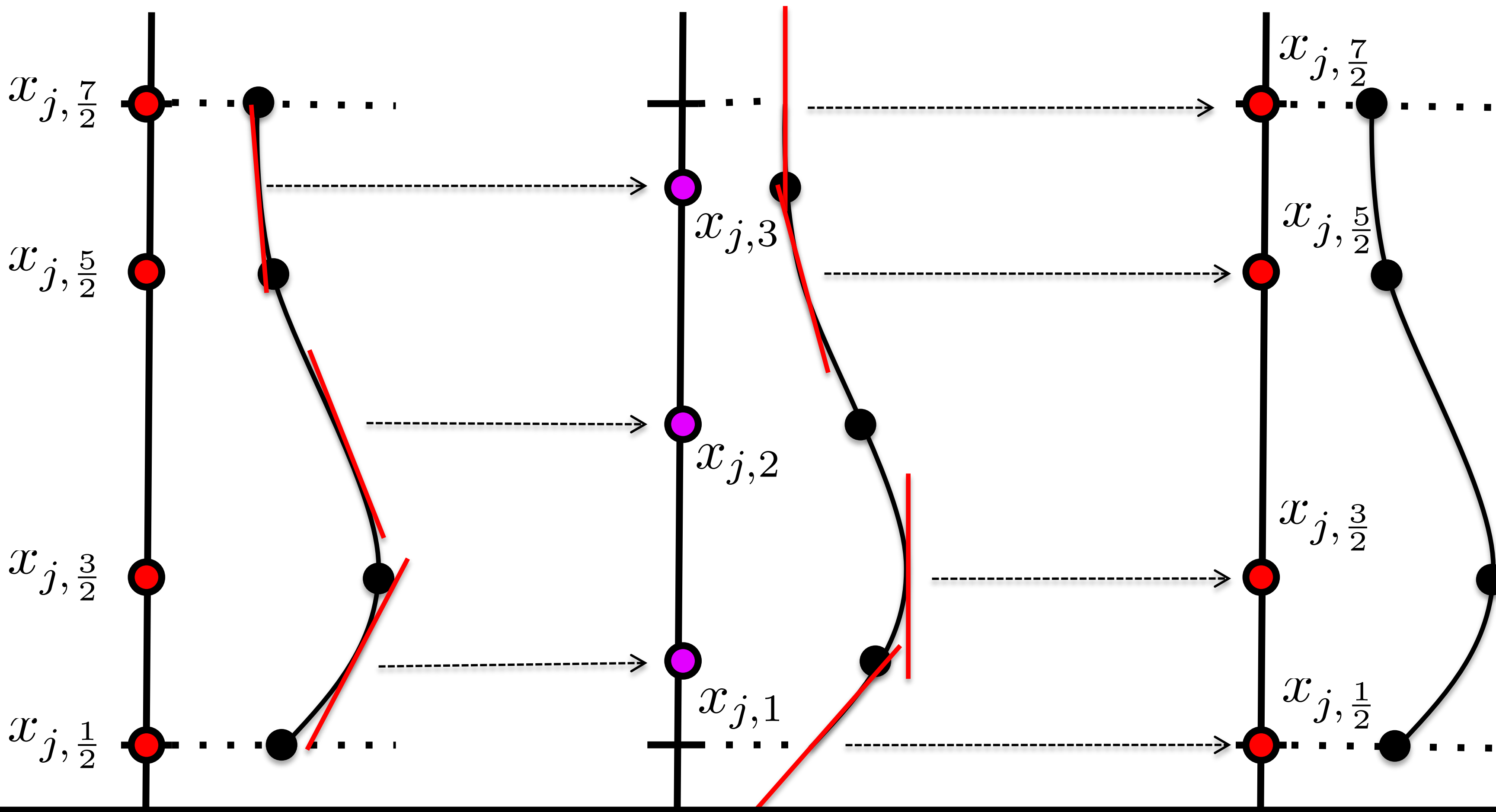
Implicit Timestepping

$(w\theta,uv\rho)^\Pi$ grid



The Hybrid Finite Element Method

Since both the spectral element method and discontinuous Galerkin method can be formulated in **differential form**, one can apply computed derivatives to nodes stored on the dual grid.



The Hybrid Finite Element Method

Hybrid finite element methods provide a natural extension of the **Charney-Phillips grid** to finite element methods.

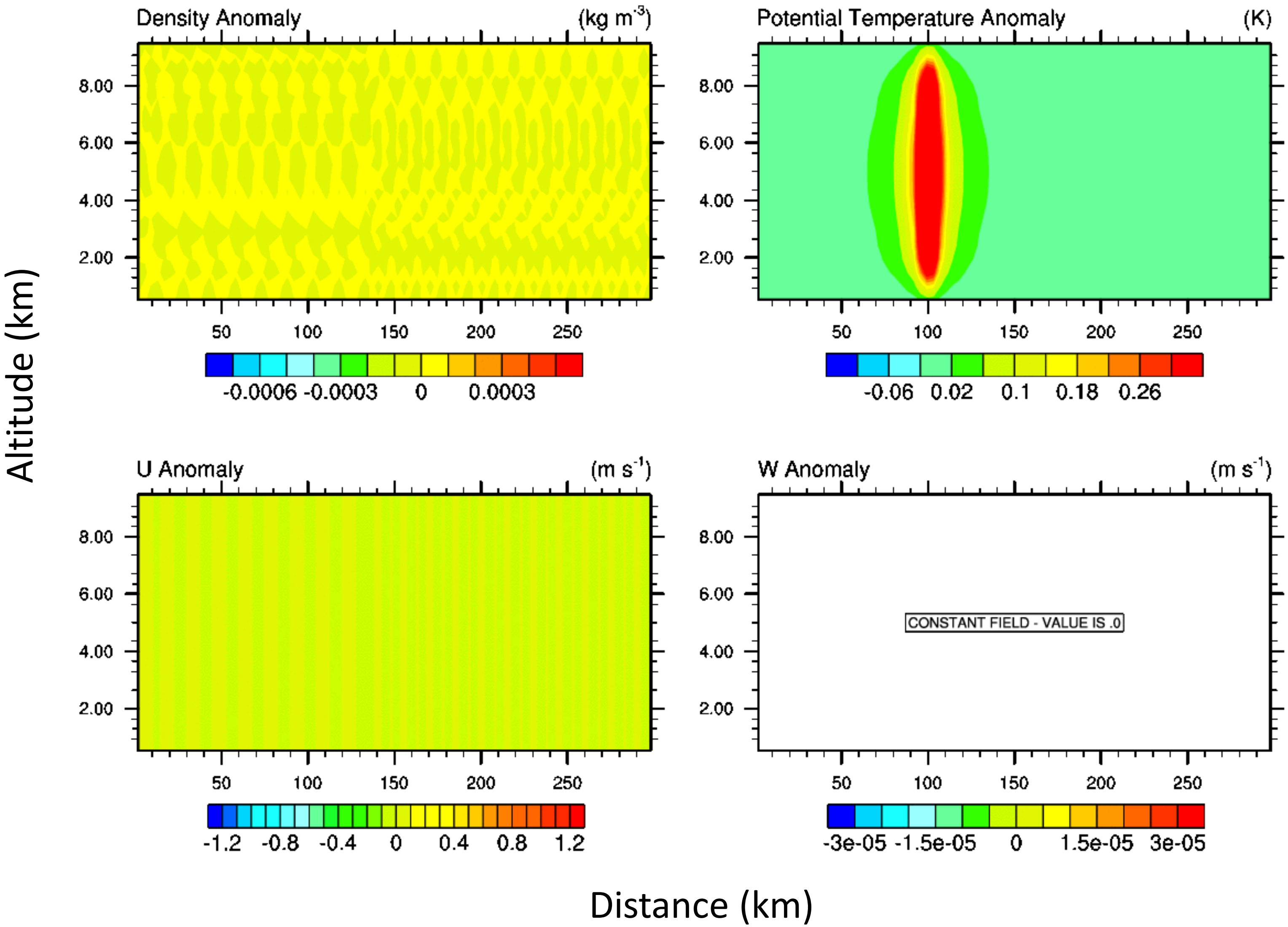
Advantage: These methods support arbitrary order of accuracy and allow for practically any choice of vertical variables. In conjunction with improved horizontal-vertical coupling, this approach can improve pressure gradient errors.

Advantage: Hydrostatic balance can be captured more effectively by a high-order reconstruction (and so does not require the use of a background reference profile in non-hydrostatic models).

Advantage: With a centered interface flux this scheme is mimetic (energy conserving) at all orders of accuracy.

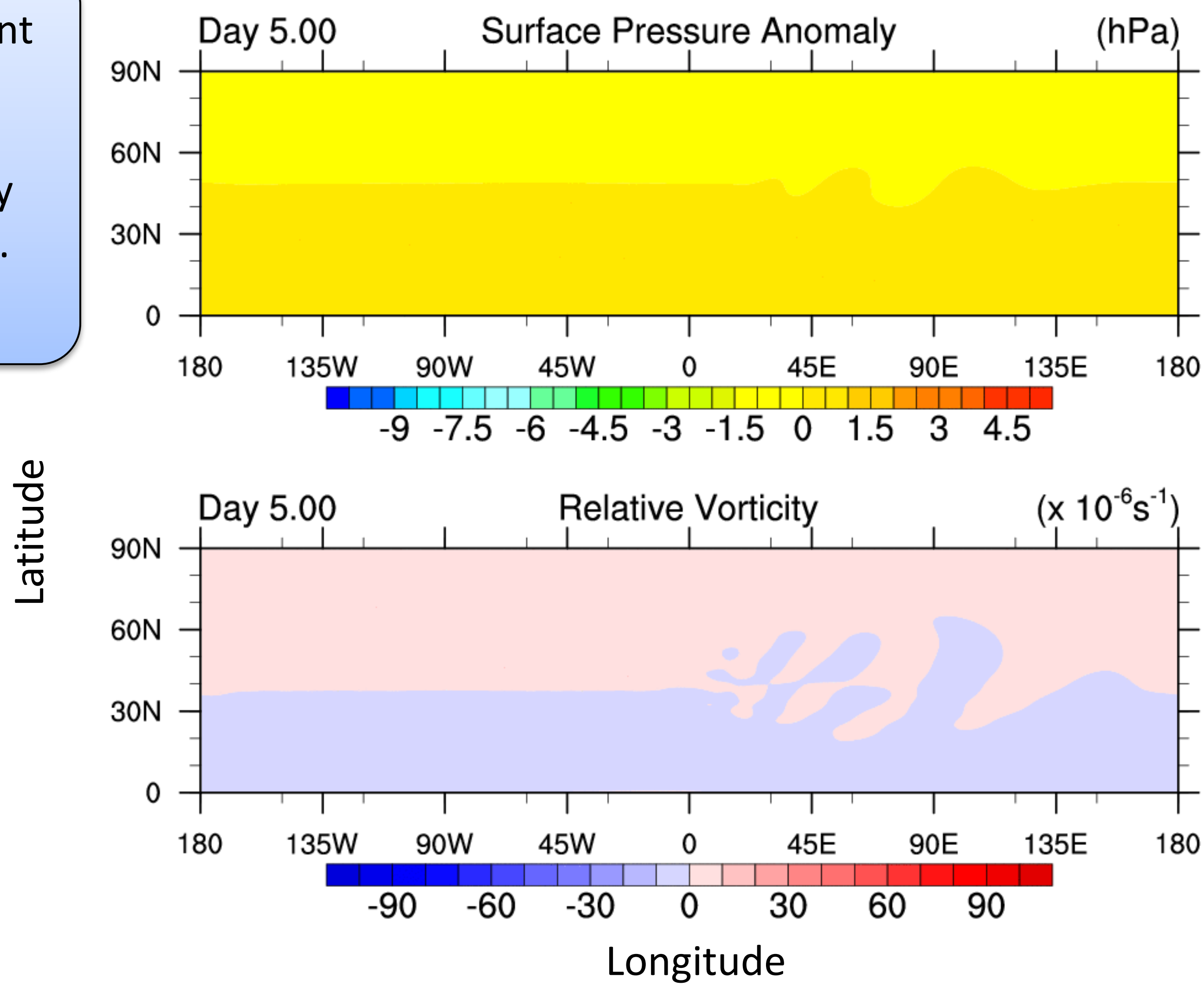
Inertia-Gravity Wave Test

Fifth-order HFEM vertical discretization, Strang splitting, no reference profile



UMJS Baroclinic Instability

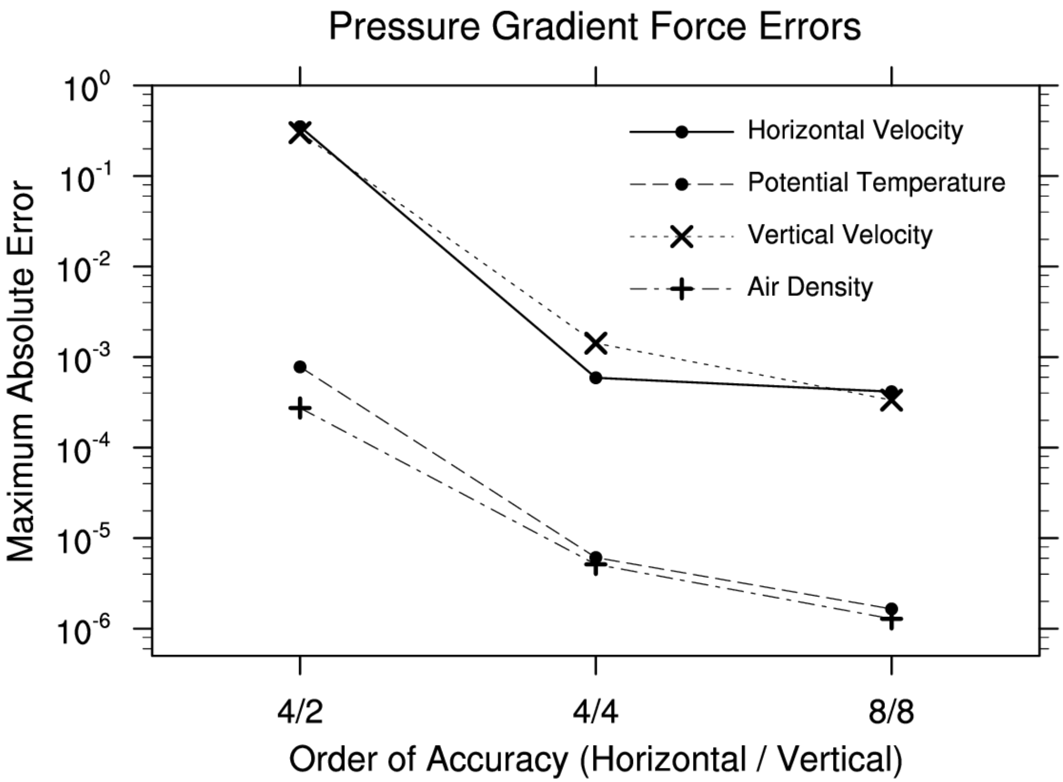
The test case is a different implementation of the Jablonowski Williamson (JW) baroclinic instability described in Ullrich et al. (2013).



Pressure Gradient Errors

As expected, the high-order treatment of the vertical also greatly improved errors due to the pressure gradient force by more than two orders of magnitude.

Additional improvement can also be obtained by combining this scheme with a high-order IMEX method for timestepping.



Next: 2D HFEM

2nd/3rd order hybrid finite-element methods in 2D. The outer "box" depicts a single finite element.

Degrees of freedom are stored at the nodes indicated, with locations dependent on the particular prognostic variable.

Also see talks by Colin Cotter, Thomas Melvin and Jemma Shipton.

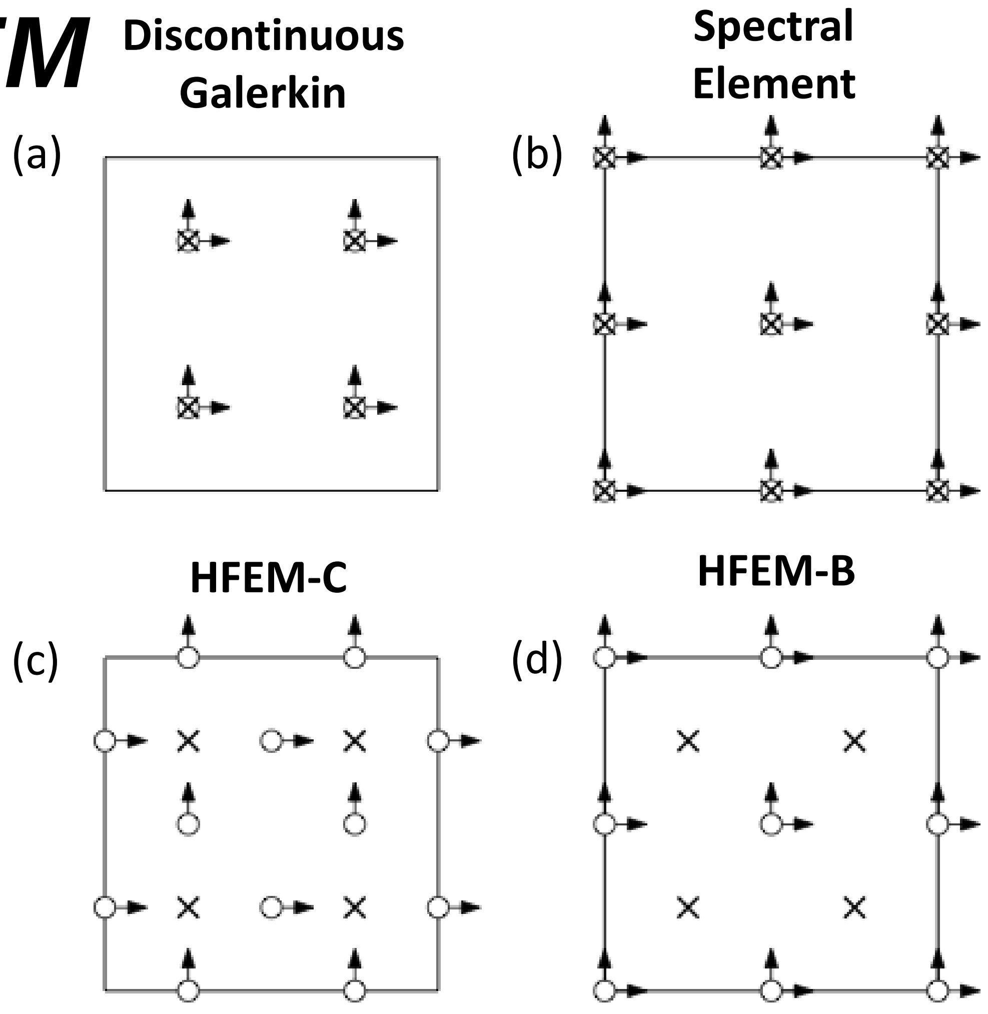


Figure: 'x' denotes the nodal location of density and potential temperature. Arrows denote locations of velocities.

Other Interests / Open Problems

Topics I would love to discuss further:

- Analysis of hybrid/mixed finite element methods and removal of the "spectral gap" via basis modification.
- Derivation of (semi-)analytical solutions to the linearized Euler equations about an isothermal background (mountain waves and inertia-gravity waves).
- Development of a vector hyperviscosity operator for discontinuous Galerkin methods.
- Conservative and monotonic remeshing between a finite-element mesh to an arbitrary finite-volume grid.
- Adaptive and static mesh refinement strategies
  (see Jared Ferguson's talk)

Thank You!

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