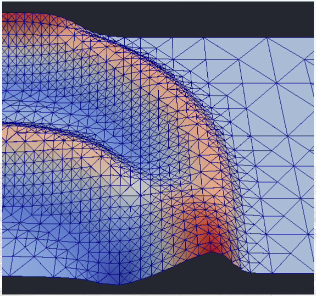
Parallel Adaptive Tsunami Modelling with Triangular Discontinuous Galerkin Schemes



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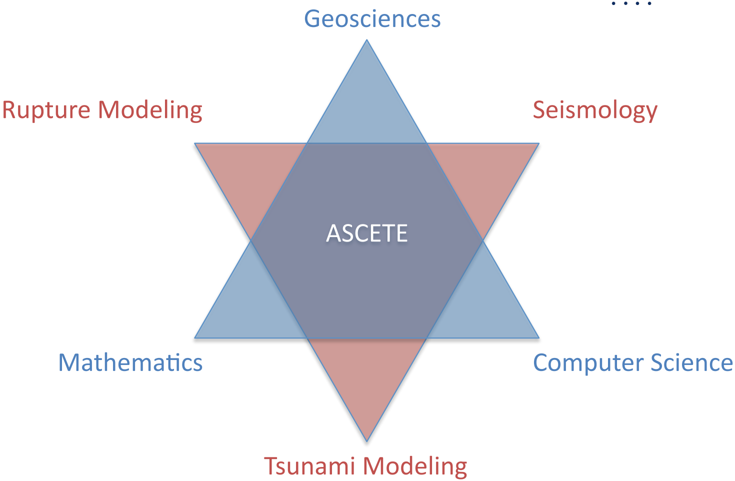
2014 PDES Workshop
“Partial Differential Equations on the Sphere”

Tsunami Modeling within ASCETE

Advanced Simulation of Coupled Earthquake and Tsunami Events

VolkswagenStiftung



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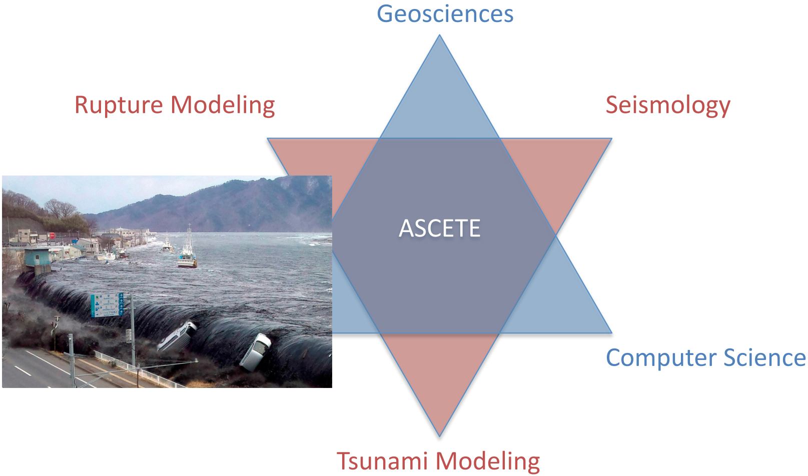
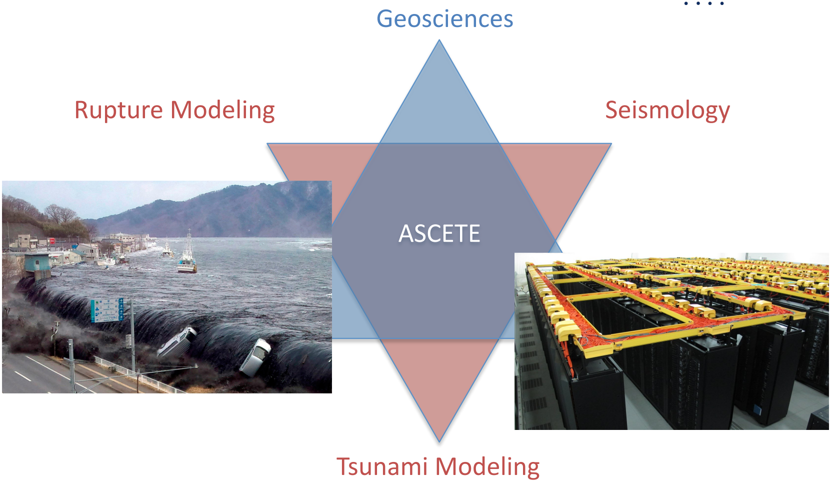


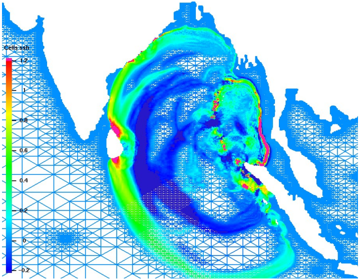
Tsunami Modeling within ASCETE

Advanced Simulation of Coupled Earthquake and Tsunami Events



A numerical algorithm for tsunami simulation should . . .

- accurately represent wave propagation and indundation at the coast,
- be well-balanced (preserves the still water steady state),
- be mass conservative,
- be robust (non-oscillatory, positive, . . .), high order of minor priority,
- be computationally efficient (small stencil, . . .).



- system of conservation laws with source term (balance laws):

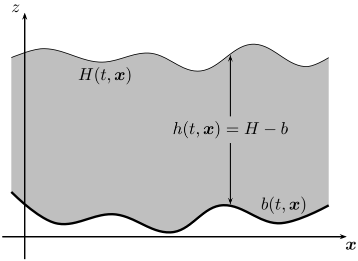
$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{U}) = \mathbf{S}(\mathbf{U}) \qquad \text{in } \Omega \times T \subset \mathbb{R}^2 \times \mathbb{R}^+$$

with $\mathbf{U} = (\phi, \phi \boldsymbol{u})^T$, $\phi = gh$, subject to initial and boundary conditions

- flux and source defined as

$$\mathbf{F}(\mathbf{U}) = \begin{pmatrix} \phi \boldsymbol{u} \\ \phi \boldsymbol{u} \circ \boldsymbol{u} + \frac{1}{2}\phi^2 \mathbf{I} \end{pmatrix}$$

$$\mathbf{S}(\mathbf{U}) = -\begin{pmatrix} 0 \\ \mathbf{S}_\mathrm{b} + \mathbf{S}_\mathrm{r} + \mathbf{S}_\mathrm{f} + \mathbf{S}_\mathrm{v} \end{pmatrix}$$



$\mathbf{S}_\mathrm{b} = \phi g \nabla b$	bathymetry
$\mathbf{S}_\mathrm{r} = f(\boldsymbol{k} \times \phi \boldsymbol{u})$	Coriolis force
$\mathbf{S}_\mathrm{f} = -\gamma^2 \boldsymbol{u} \lVert \boldsymbol{u} \rVert / \phi^{1/3}$	bottom friction
$\mathbf{S}_\mathrm{v} = -\nu \nabla(\phi \boldsymbol{u})$	eddy viscosity

- Galerkin ansatz (residual orthogonal to some function space V):

$$\int_\Omega \left(\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{U}) - \mathbf{S}(\mathbf{U}) \right) \psi \, \mathrm{d}\boldsymbol{x} = 0 \quad \forall \psi \in V$$

- weak DG formulation (integration by parts):

$$\int_{T_\omega} \left(\frac{\partial \mathbf{U}_h}{\partial t} - \mathbf{F}_h \cdot \nabla - \mathbf{S}_h \right) \psi_\omega \, \mathrm{d}\boldsymbol{x} = - \int_{\partial T_\omega} \psi_\omega \mathbf{F}_h^* \cdot \boldsymbol{n} \, \mathrm{d}\sigma$$

Representation on reference element:

- nodal approach based on electrostatics / Fekete points
- local solution represented by Lagrange polynomials $\psi_i \in V_h$ of degree p

$$\mathbf{U}(\boldsymbol{\xi}, t) = \sum_{i=1}^{r} \mathbf{U}(\boldsymbol{\xi}_i, t) \psi_i(\boldsymbol{\xi}) = \sum_{i=1}^{r} \tilde{\mathbf{U}}_i(t) \psi_i(\boldsymbol{\xi})$$

following GIRALDO ET AL. [2002], GIRALDO [2006]

SSP Multistage scheme:

$$\mathbf{U}_h^{(0)} = \mathbf{U}_h^n$$

$$\mathbf{U}_h^{(i)} = \Lambda\Pi_h \left\{ \sum_{l=0}^{i-1} \alpha_{il}\mathbf{U}_h^{(l)} + \beta_{il}\Delta t^n \mathbf{H}_h(\mathbf{U}_h^{(l)}) \right\} \quad \text{for } i = 1 \ldots k+1$$

$$\mathbf{U}_h^{n+1} = \mathbf{U}_h^{(k+1)}$$

Intermediate solution limited with $\Lambda\Pi_h$ after each stage:

- TVB corrected minmod function [Cockburn and Shu, 1998]:

$$\bar{m}(u_{i,x}, \bar{u}_{i+1} - \bar{u}_i, \bar{u}_i - \bar{u}_{i-1})$$

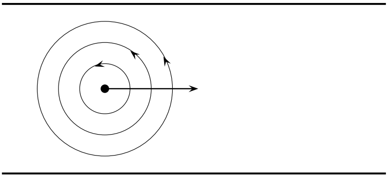
- limiting in conservative (h, hu) vs. hydrostatic variables $(h + b, hu)$
- positivity preservation [Xing et al., 2010]:

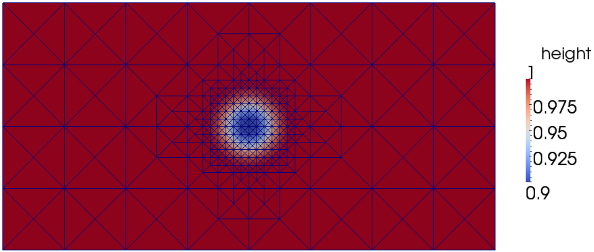
$$\tilde{U}^n(x,y) = \theta\left(U^n(x,y) - \overline{U^n}\right) + \overline{U^n} \quad \text{with } \theta = \min\left(1, \frac{\overline{\phi^n}}{\overline{\phi^n} - \min_x\{\phi^n(x)\}}\right)$$

2D Test Case

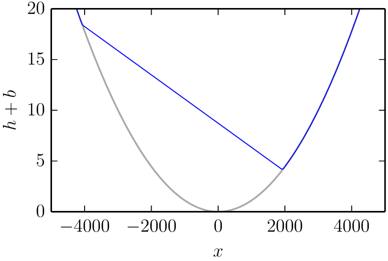
Quasi Stationary Vortex Advected by Constant Background Flow

- using grid library `amatos` (conforming elements, refinement by bisection)
- linear elements, no limiting
- simple gradient based error indicator





- domain $[-5000, 5000]$ with $b(x) = 10(x/3000)^2$
- 200 cells, $\Delta t = 1.0$, $\mathrm{CFL} \approx 0.2$
- limiting in (h, hu)



- analytical solution by Thacker (1981):

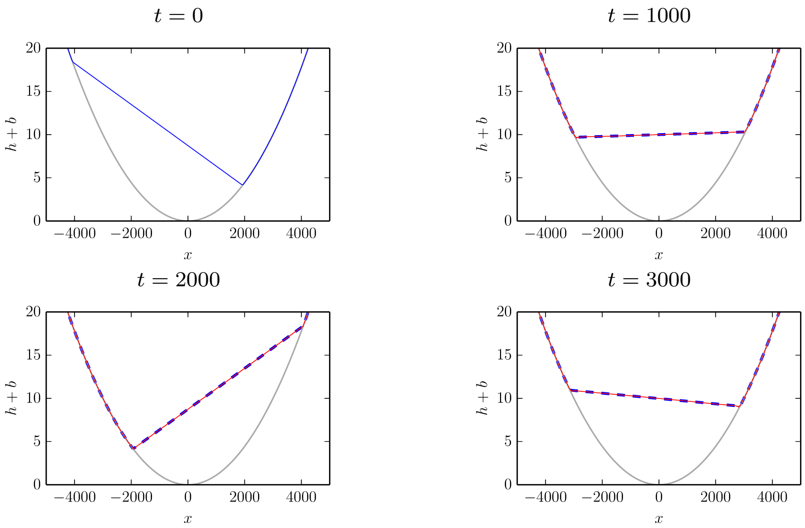
$$h(x,t) + b(x) = h_0 - \frac{B^2}{4g}\left(1 + \cos(2\omega t)\right) - \frac{Bx}{2a}\sqrt{\frac{8h_0}{g}}\cos\omega t,$$

$\omega = \sqrt{2gh_0}/a$, $B = 5$

- initial momentum at $t = 0$ is set to zero

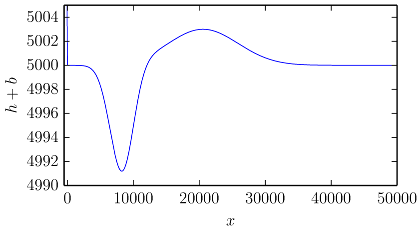
1D Results

Oscillatory Flow in a Parabolic Bowl

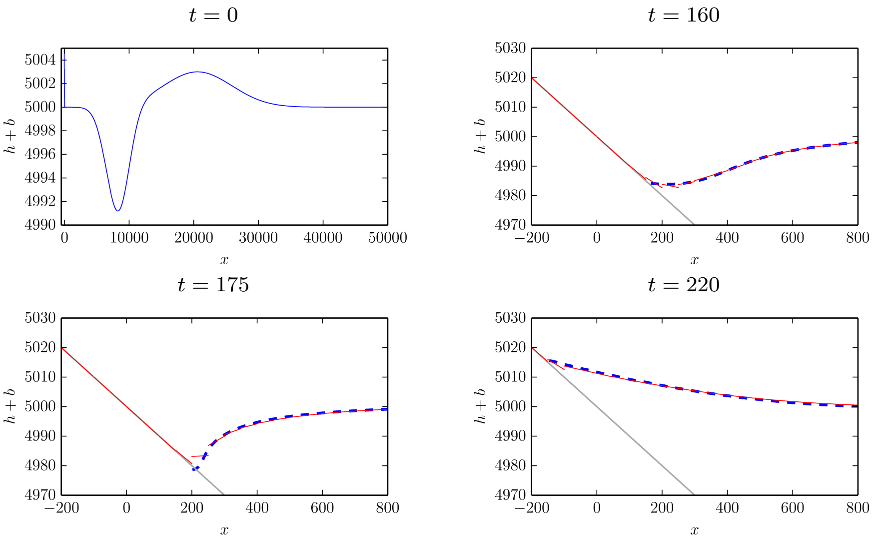


Benchmark problem 1 from “The Third International Workshop on Long-Wave Runup Models” (2004).
`http://isec.nacse.org/workshop/2004_cornell/bmark1.html`

- uniformly sloping beach with $b(x) = 5000 - 0.1x$
- prescribed initial surface elevation and momentum with $(hu)(x, 0) \equiv 0$
- domain $[-500; 50000]$, 1010 cells
- $\Delta t = 0.05$, $\mathrm{CFL} \approx 0.2$
- limiting in (h, hu)



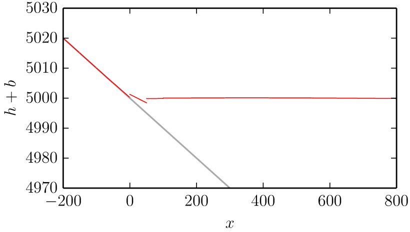
1D Results
Tsunami Runup onto a Sloping Beach

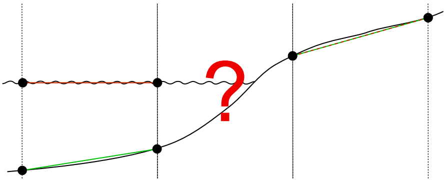


Wellbalancing with Wetting and Drying

How to Discretize a Semi-dry Cell?

Assuming a fixed spatial grid (no moving grid points), piecewise linear DG discretization:

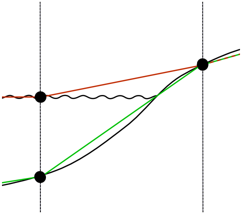




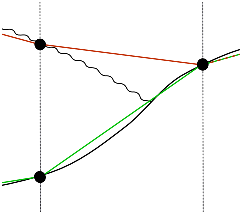
- Tsunami Runup at $t = 50$
- limiting in (h, hu)

Well-balancing with wetting/drying:

- distinguish between “flood type” and “dam-break type” cells
- neglect gradient in surface elevation in flood-type cells
- inner cell redistribution of tendencies to wet nodes in flood-type cells
- limiting only in fully wet cells, adjustment of negative heights

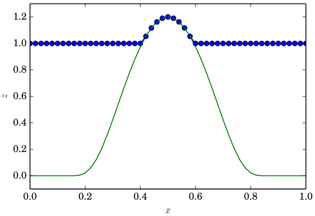


flood type

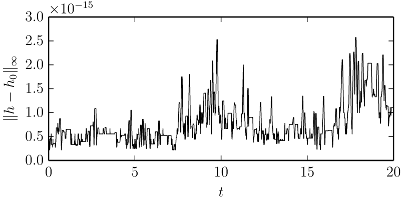


dam-break type

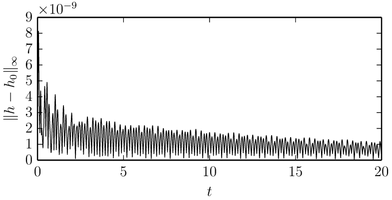
- domain $[0, 1]$ with periodic boundary conditions
- still water $u \equiv 0$
- 50 cells, $\Delta t = 0.002$, $\mathrm{CFL} \approx 0.3$, $t_{\max} = 20$
- limiting in $(h + b, hu)$



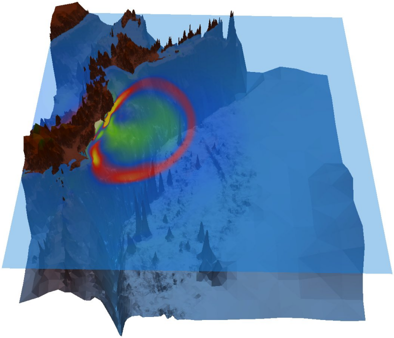
Test initialized with zero momentum field:

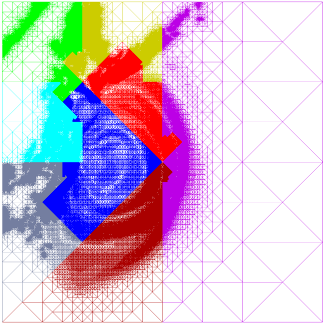


...with random deviations in the momentum field of the order 10^{-8}:



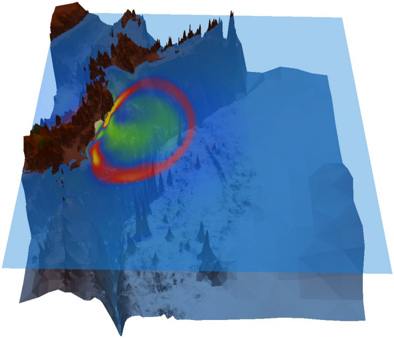
Parallel AMR for Triangular Grids
Challenges for Grid Refinement and Coarsening

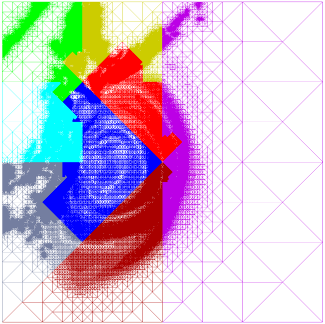




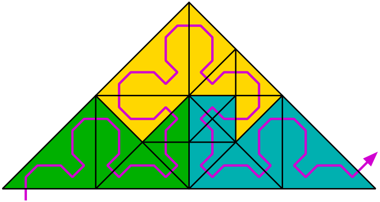
- strong, dynamically adaptive refinement (to capture solution and geometrical details etc.)
- frequent re-meshing of large parts of the grid
- substantial change of problem size during simulation

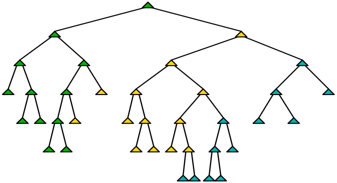
Parallel AMR for Triangular Grids
Challenges for Parallelisation





- trend to multi-/manycore; multiple layers of parallelism
- dynamic load balancing due to remeshing and varying computational load
- goal: retain locality properties (partitioning)

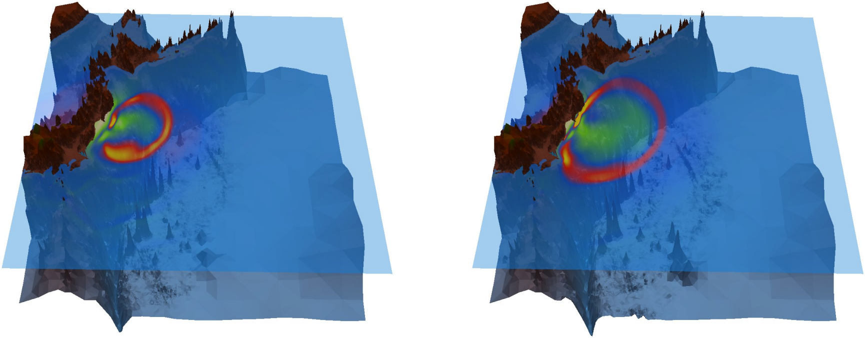




- recursively structured triangular grids (newest vertex bisection)
- fully adaptive grid described by a corresponding refinement tree
- element orders (tree and grid cells) defined by Sierpinski space-filling curves
- minimum memory requirements
  $\rightarrow$ triangle strips as data structure
- exploit locality properties for cache and parallel efficiency

Test Scenario

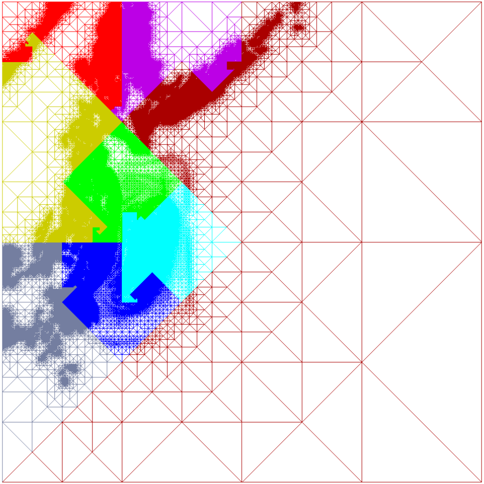
Adaptive Tsunami Simulation with Finite Volumes



- basic shallow water model, Finite Volume discretisation
- augmented Riemann solver (George, 08) provided by GeoClaw
- dynamically adaptive grid with up to 100 Mio grid cells
- refinement/coarsening and load balancing after each time step

Test Scenario

Adaptive Tsunami Simulation with Finite Volumes



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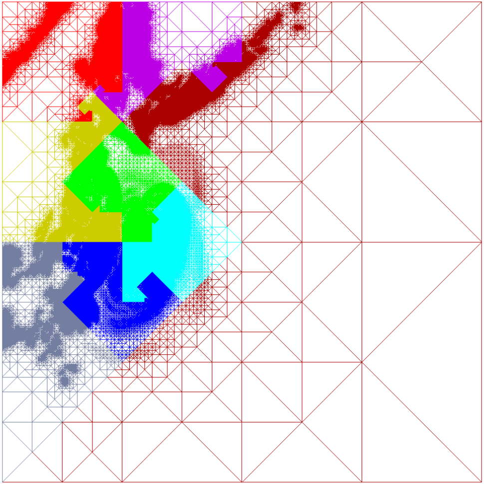
Test Scenario

Adaptive Tsunami Simulation with Finite Volumes



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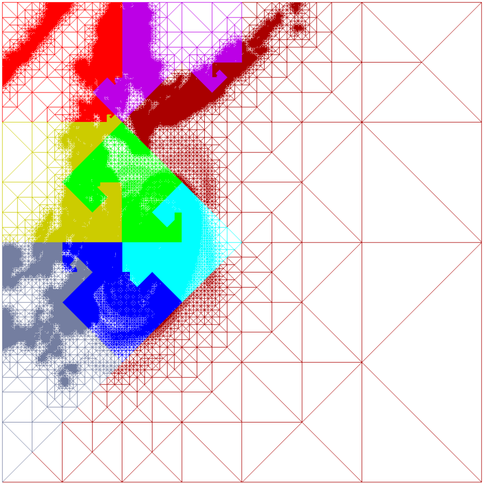
Test Scenario

Adaptive Tsunami Simulation with Finite Volumes



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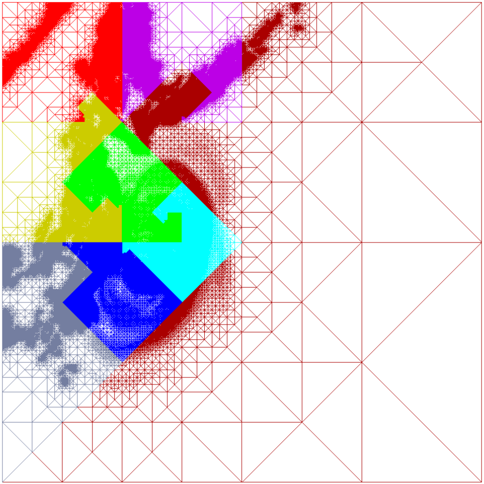
Test Scenario

Adaptive Tsunami Simulation with Finite Volumes



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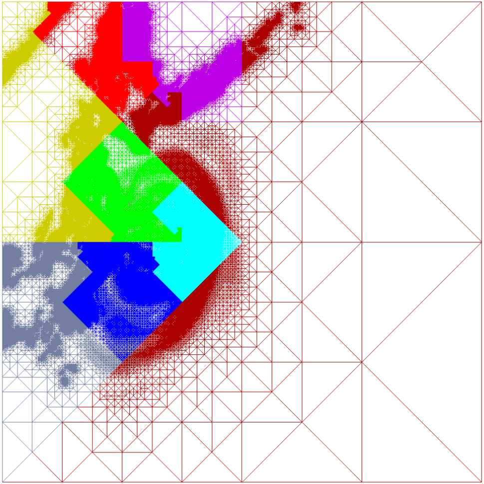
Test Scenario

Adaptive Tsunami Simulation with Finite Volumes



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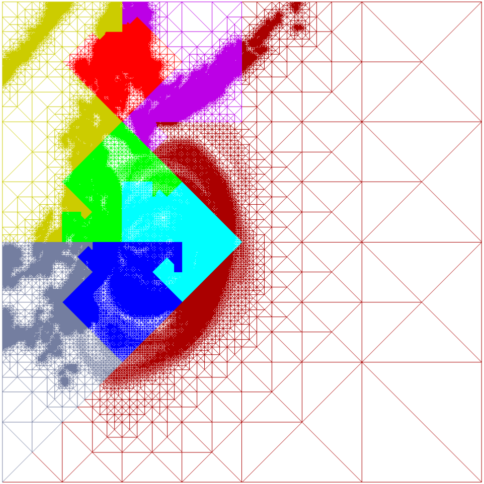

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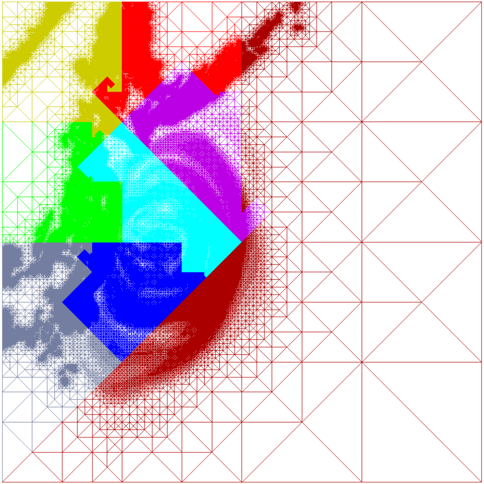
Test Scenario

Adaptive Tsunami Simulation with Finite Volumes



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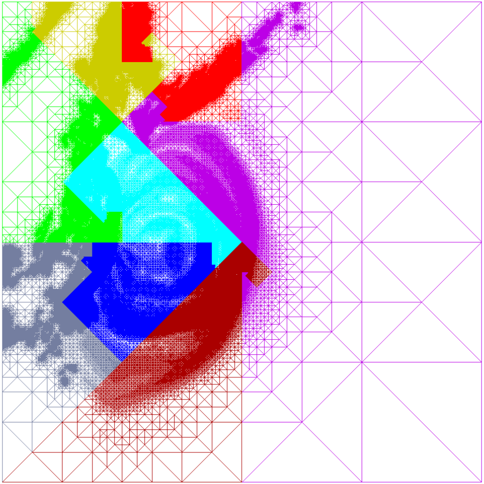

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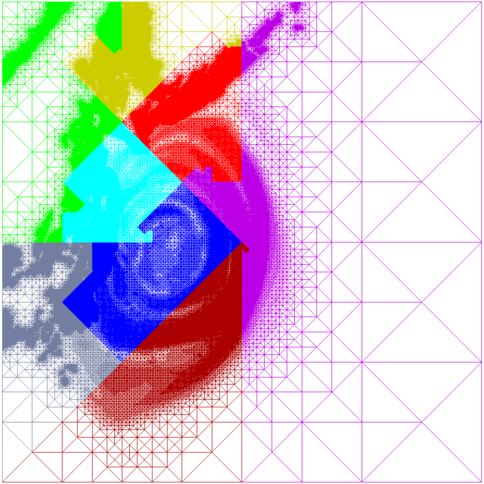
Test Scenario

Adaptive Tsunami Simulation with Finite Volumes



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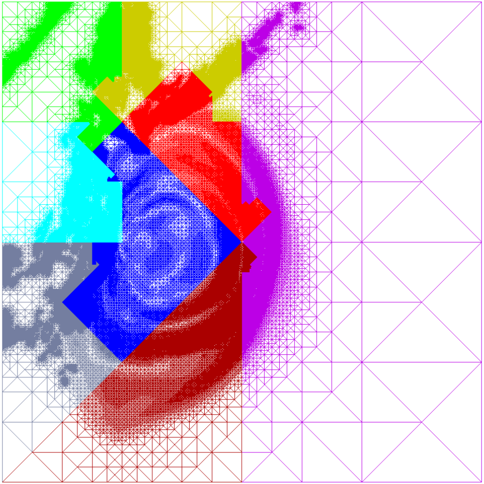
Test Scenario

Adaptive Tsunami Simulation with Finite Volumes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Summary

- tsunami modeling on adaptive triangular grids
- discretization by discontinuous Galerkin method
- wetting & drying for DG (positivity vs. well-balancing)
- parallel adaptive triangular grids (Sierpinski SFC)
- locality properties for cache and parallel efficiency

Outlook

- further development of wetting/drying treatment (positivity AND well-balancing)
- extension to 2D, realistic tsunami simulations
- implementation into parallel framework for HPC