Curl-free pressure gradients over orography in a solution of the fully compressible Euler equations with long time-steps

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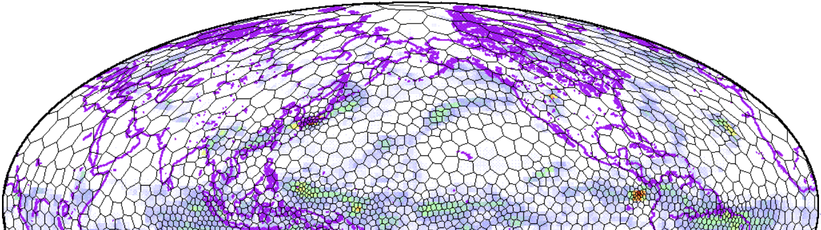
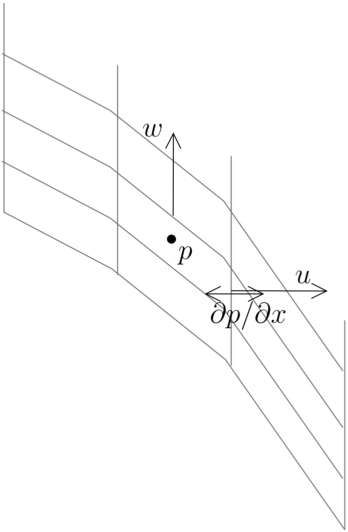


Motivation

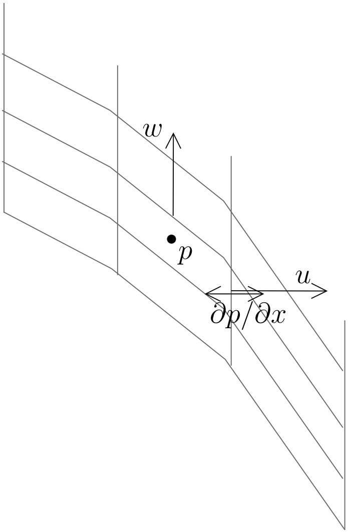
- Mimetic properties over orography
  - using mimetic horizontal discretisation in the vertical
- Long time-steps
  - suitable for massively parallel
  - suitable for unstructured grids
  - simpler than SISL (semi-implicit, semi-Lagrangian)

Representing Mountains



For numerous reasons in meteorology the cells should line up in columns

Representing Mountains



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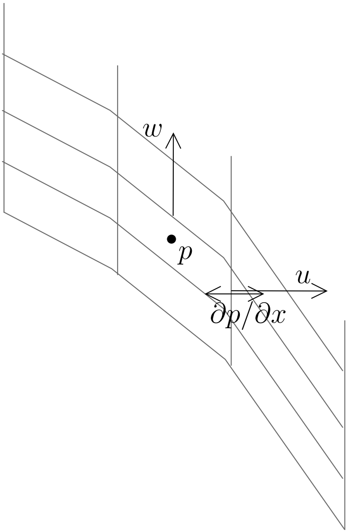

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Usual approach: orthogonal prognostic velocity variables u, v, w in horizontal and vertical directions

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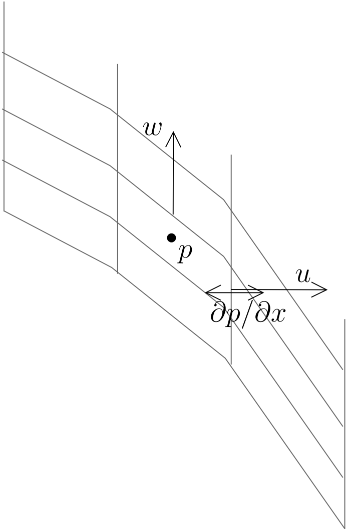

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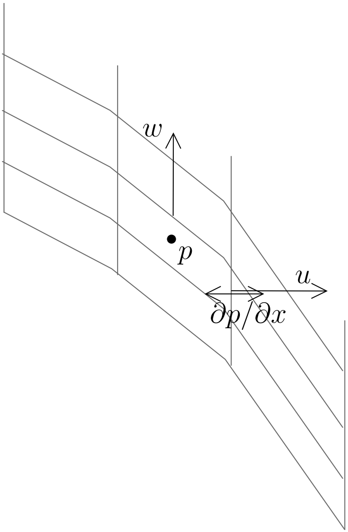

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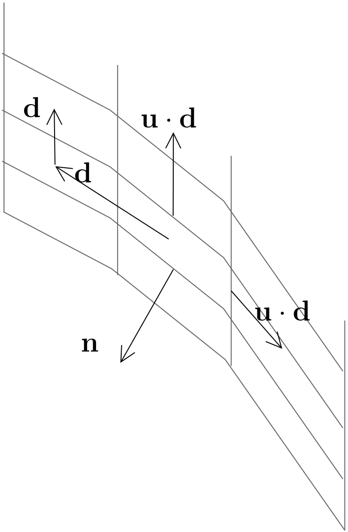
Alternative: non-orthogonal prognostic variables (covariant)



Following horizontal discretisations on non-orthogonal grids:

Prognostic variables: $\mathbf{u}\cdot\hat{\mathbf{d}}$

where $\mathbf{d}_f = \mathbf{x}_i - \mathbf{x}_j$

→ curl free pressure gradients

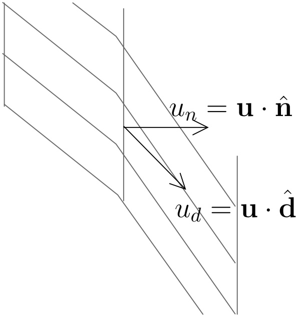
→ no spurious generation of vorticity

Non-orthogonal prognostic variables (covariant)

Need mass flux $u_n = \mathbf{u} \cdot \hat{\mathbf{n}}$ in continuity equation
Requires operator H from space of all u_ds to space of all u_ns.

$$u_n = H u_d$$

(following Thuburn, Dubos, Cotter, 2014)



- First reconstruct full velocity at face f from $u_{d'}$ at the surrounding faces, f':

$$\mathbf{u} = T_i^{-1} \sum_{f'} \mathbf{d}_{f'} A_{f'} u_{d'}$$

$$\text{where } \; T = \sum_{f'} \hat{\mathbf{d}}_{f'} \hat{\mathbf{d}}_{f'}^T A_{f'}$$

Least squares fit which reconstructs a uniform velocity field

- Next take component in direction $\mathbf{n}$ and correct the component in direction $\mathbf{d}$ so that the result is exact on an orthogonal face:

$$u_n = \mathbf{u} \cdot \hat{\mathbf{n}} + \left(u_d - \mathbf{u} \cdot \hat{\mathbf{d}} \right) \left(\hat{\mathbf{n}} \cdot \hat{\mathbf{d}} \right)$$

The resulting H is asymmetric which violates energy conservation

Results

Resting stratified atmosphere over a steep mountain

- should remain stationary
- potential temperature contours should remain horizontal

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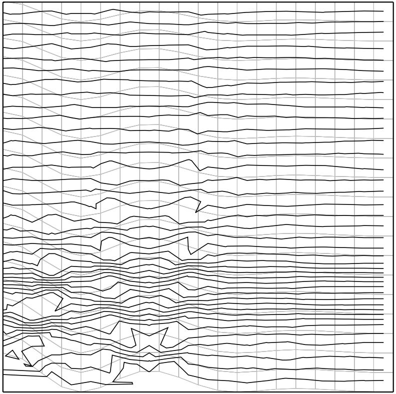
$\partial p/\partial x$ version, implicit gravity waves
Δt=100s. Maximum $N\Delta t$=2



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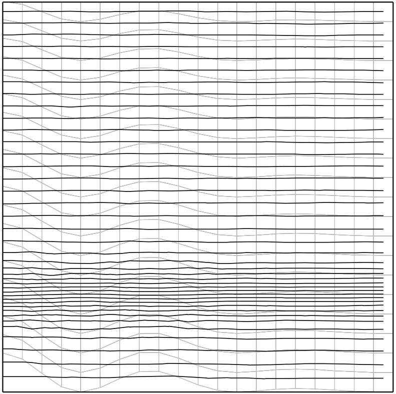
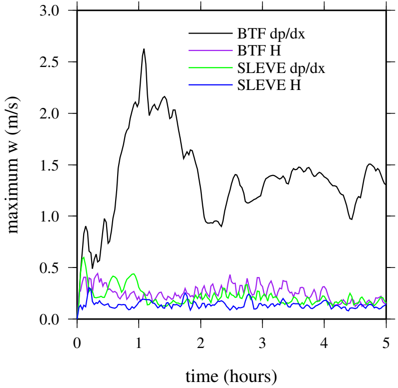


Maximum Spurious Velocity



New models

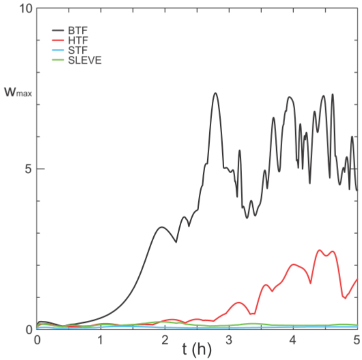
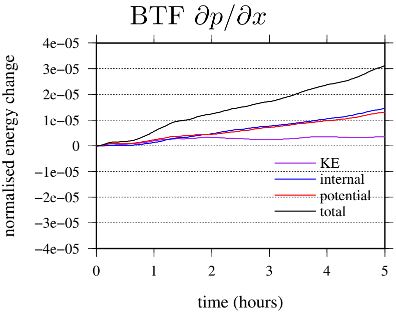


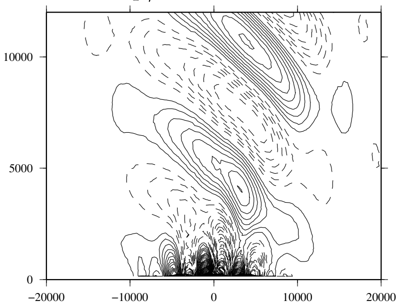
Figure 4 from Klemp [2011]

Energy Conservation



Gravity-waves over orography Δt=40s, $u\Delta t/\Delta x \approx 1$, w every .05m/s

$\partial p/\partial x$ version



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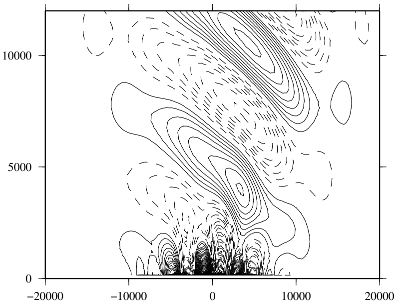
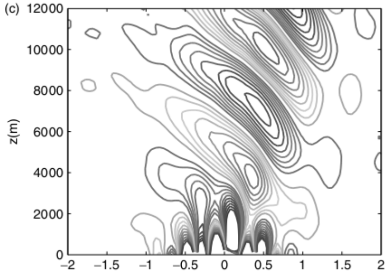
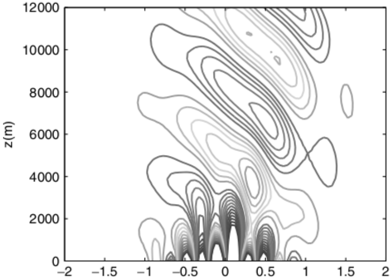


Met Office [Melvin et al., 2010]



Linear analytic solution



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Neither of these are necessary:
Starting from the Euler equations:

Momentum	$\partial \mathbf{u}/\partial t + \mathbf{u}\cdot\nabla\mathbf{u} + 2\Omega\times\mathbf{u} = \mathbf{g} - c_p\theta\nabla\Pi$
Continuity	$\partial\rho/\partial t + \mathbf{u}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{u} = 0$
Potential temperature	$\partial\theta/\partial t + \mathbf{u}\cdot\nabla\theta = 0$
State	$\Pi^{\frac{1-\kappa}{\kappa}} = R\rho\theta/p_0$

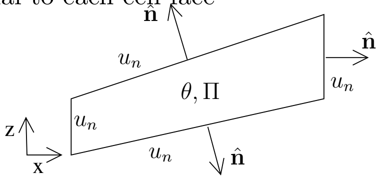
where potential temperature $\theta = T\left(p_0/p\right)^{\kappa}$
Exner function of pressure $\Pi = \left(p/p_0\right)^{\kappa}$

In order to treat acoustic and gravity wave implicitly, these must ALL be combined to form a linearised equation for Π

Semi-implicit acoustic and gravity waves

Prognostic variables:

$u_n = \mathbf{u} \cdot \hat{\mathbf{n}}$	velocity normal to each cell face
θ	at cell centre
Π	at cell centre



Overview:

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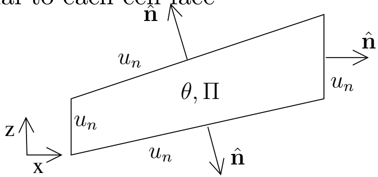

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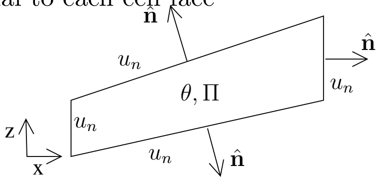

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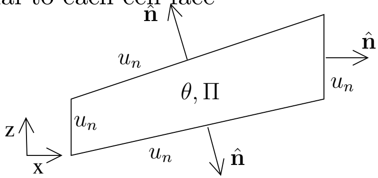

Overview:

- Substitute θ equation into momentum equation
- Substitute u_n into continuity equation
- Use equation of state to replace ρ with Π in continuity equation:
  $\rightarrow$ Helmholtz equation for Π

Substitute θ into momentum equation

Rearrange θ equation to give θ^{n+1} in terms of u_n^{n+1} (1^{st} order in time for brevity):

$$\partial\theta/\partial t + \mathbf{u}\cdot\nabla\theta = 0 \;\rightarrow\; \theta^{n+1} = \theta^n - \Delta t\;\mathbf{u}\cdot\nabla\theta^n$$

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$$\text{where}\quad \begin{aligned}\mathbf{u}^\perp &= \mathbf{u} - (\mathbf{u}\cdot\hat{\mathbf{n}})\,\hat{\mathbf{n}} \\ \nabla_n\theta &= (\nabla\theta)\cdot\hat{\mathbf{n}}\end{aligned}$$

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and write as:

$$u_n^{n+1} = G\left(u' - \Delta t c_p \theta' \nabla_n \Pi^{n+1}\right)$$

Substitute u_n^{n+1} into the continuity equation to get ρ^{n+1} in term of Π^{n+1}

Final Construction of Helmholtz equation

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where $\Psi = \left(\rho^{\ell}\right)^{\frac{2\kappa-1}{\kappa-1}} (R\theta/p_0)^{\frac{\kappa}{\kappa-1}} \approx (p_0/R)^{0.4} \rho^{0.6}/\theta^{0.4}$

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Solve to find Π^{n+1} in terms of Π^n then back substitute to get ρ^{n+1}, u_n^{n+1} and θ^{n+1}.
This is VERY convergent and allows long time steps w.r.t. gravity and acoustic wave speeds ... but what about advection ...

Sub time-steps for advection

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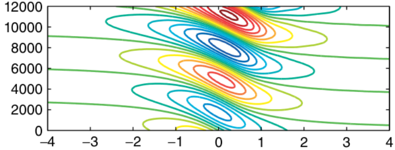
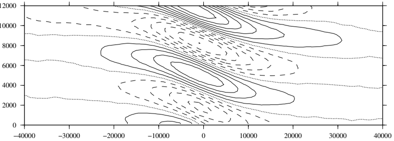
- Linear stability analysis does not reveal any time-step restrictions (not shown)

Hydrostatic Mountain Waves (To test long time-steps)

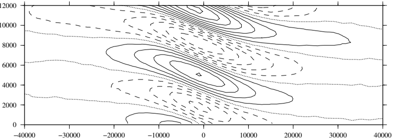
Linear analytic solution



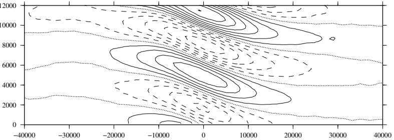
Implicit gw, $\Delta t = 20\text{s}$, $Co = 0.2$, $N\Delta t = 0.4$



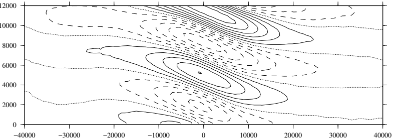
Explicit gw, $\Delta t = 20\text{s}$, $Co = 0.2$, $N\Delta t = 0.4$



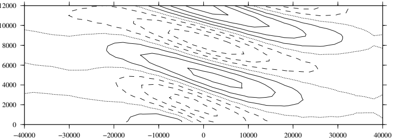
Implicit gw, $\Delta t = 100\text{s}$, $Co = 1$, $N\Delta t = 2$



Implicit gw, $\Delta t = 200\text{s}$, $Co = 2$, $N\Delta t = 4$



Resolution$\div 2$, $\Delta t = 500\text{s}$, $Co = 2.5$, $N\Delta t = 10$



Stable at long time-steps but accuracy is lost because θ advection is implicit rather than sub-stepped - needs sorting

Conclusions

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  - curl-free pressure gradients
  - better stable stratification over orography
- Semi-implicit treatment of acoustic and gravity waves
  - without treating z-direction differently
  - without an explicitly defined reference profile
  - suitable for unstructured in the vertical
  - allows time-step independent of stratification
- Sub time-steps for advection
  - allows arbitrary long time-steps for the implicit terms

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T. Melvin, M. Dubal, N. Wood, A. Staniforth, and M. Zerroukat. An inherently mass-conserving semi-implicit semi-Lagrangian discretisation of the nonhydrostatic vertical slice equations. *Quart. J. Roy. Meteor. Soc.*, 137:799–814, 2010.