A new covariant form of the equations of geophysical fluid dynamics and their structure-preserving discretization

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Abstract

The accuracy of computational models used to simulate, for instance the flow features of atmosphere and ocean, crucially depends on the sets of analytical equations that describe the physical phenomena to be simulated, and on how these analytical equations are discretized to make them suitable for computational calculations. The discrete model should hereby inherit certain conservation properties from the analytical equations, such as mass and energy conservation and the preservation of certain flow features. Already the formulation of the analytical equations often provides a guideline to find the corresponding set of discrete equations.

I introduce a new formulation of the analytical equations of geophysical fluid dynamics (GFD), which provides a methodology to find structure-preserving discretized equations for computational calculations. The resulting discrete model is structure-preserving in the sense that the calculated approximated solutions are mass and energy conserving and that the discrete velocity fields consist of divergence-free and rotation-free parts, analogously to the analytical fields. The latter property is important to preserve certain flow features also in the discrete case and is often refer to as Helmholtz decomposition.

The new formulation consists in a separation of the momentum and continuity equations into a metric-free (topological) and a metric-dependent part. In the metric-free part, only neighborhood relationships are taken into account. This allows, for instance, to indicate the boundary of an area or to find its neighboring regions. The information about the metric structure, such as length, volume, angle, etc., is confined to the metric-dependent equations. This splitting of the equations has been enabled by introducing auxiliary prognostic variables and by representing all variables by differential forms instead of vector fields. The latter fact makes these equations covariant, i.e. their form is invariant under coordinate transformations. This new structure reveals important geometrical features of the equations of GFD. For instance, it illustrates how the metric-free momentum and continuity equations geometrically interact and how they are connected by the metric-dependent equations. Moreover, it shows that the quantities of interest, such as velocity and vorticity, are more adequately described by differential forms than vector fields.

Based on this split formulation, I develop a systematic discretization approach, using tool of discrete differential geometry, which leads to discrete model equations that automatically preserve important conservation properties. I illustrate this methodology on the linear shallow-water equations, for which I derive on arbitrarily structured C-grids a finite difference approximation that conserves mass and energy, while preserving the Helmholtz decomposition. Because computational models that do not conserve such properties usually require additional stabilizations, which may badly impact on the model results, this structure-preserving discretization approach helps to simulate geophysical fluid flows more accurately.