Three Topics on Normal Modes: Barotropic Equatorial Trapping and the Effective Lamb's Parameter, Kelvin Solitons and Corner Waves and Hough Eigenvalue Point Clouds

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FIRST THEME: Equatorial Trapping is Controlled by BOTH Lamb's Parameter and Zonal Wavenumber

$$\epsilon \equiv \frac{4\Omega a^2}{gH} \text{ [LAMB'S PARAMETER]}$$

$\Omega = 2\pi/84,600\,\text{s},\qquad a = \text{earth's radius}$

$g = 9.8\text{m/s} \qquad H = \text{equivalent depth}$

$s \equiv$ LONGITUDINAL WAVENUMBER, an INTEGER

Equatorial Beta-Plane: Asymptotic Approximation by Hermite Functions

- Orthodoxy: $v_n \sim \psi_n(\epsilon^{1/4}\mu)$, $\psi_n(y) = \exp(-[1/2]y^2)H_n(y)$

- Boyd (*J. Atmos. Sci.*, 1985) argued that for Rossby waves,

$$\boxed{\epsilon \rightarrow \epsilon + s^2}$$

- Boyd & Zhou (*J. Atmos. Sci.*, 2008) extended to Kelvin waves
- Argument applies to SPHERICAL HARMONICS & PROLATE SPHEROIDAL FUNCTIONS as well as HOUGH FUNCTIONS

Prolate Spheroidal Illustration

$$\frac{d^2\psi}{d\phi^2} - \frac{d\psi}{d\phi} + \left\{\chi_n - \frac{s^2}{\cos^2(\phi)} - \epsilon\sin(\phi)^2\right\}\psi = 0$$

$\Downarrow \cos(\phi) \to 1,\ \sin(\phi) \to \phi \Downarrow$

$$\frac{d^2 v}{d\phi^2} + \left\{\chi_n - s^2 - \epsilon\phi^2\right\} v = 0$$

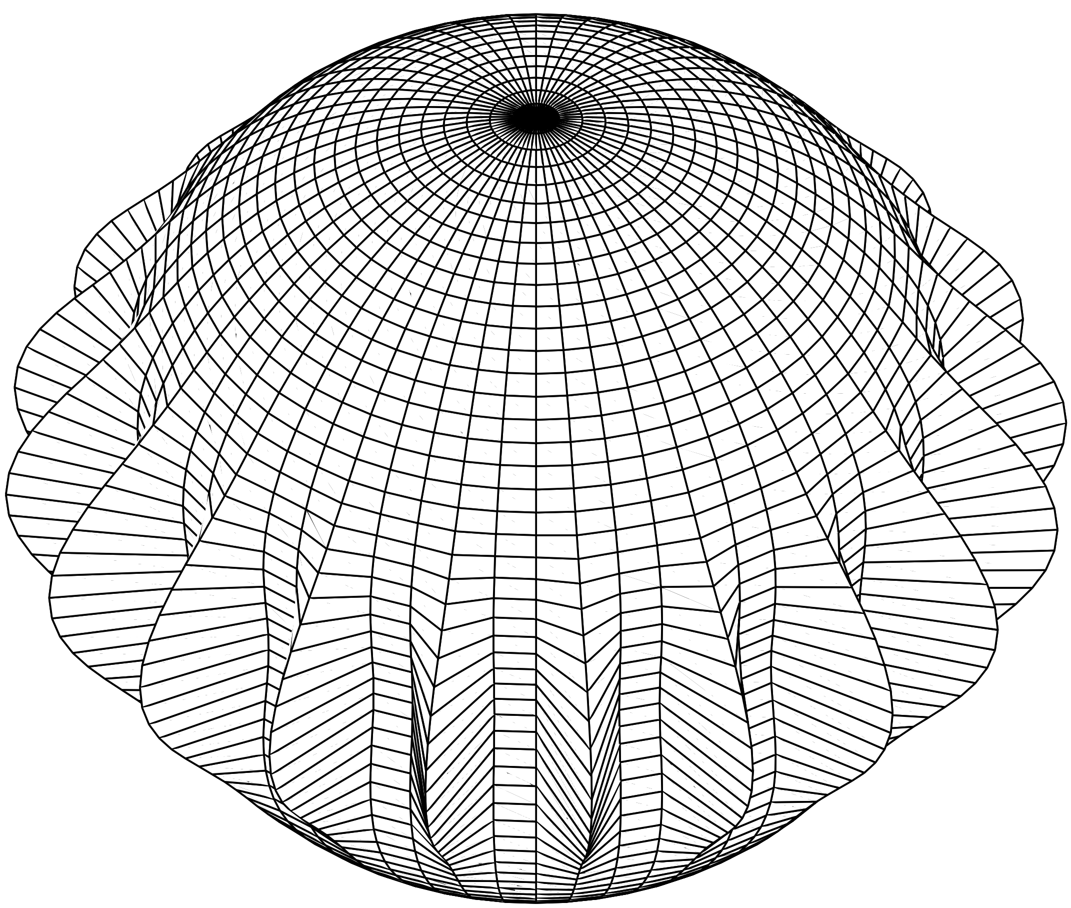
$$\begin{aligned} 1/\cos(\phi)^2 &= 1/\left\{1 - \sin^2(\phi)\right\} \\ &= 1 + \sin^2(\phi) + O(\sin^4(\phi) \\ &= 1 + \phi^2 + O(\phi^4) \end{aligned}$$

$$\begin{aligned} &-\frac{s^2}{\cos^2(\phi)} - \epsilon\sin(\phi)^2 \\ \Rightarrow\ &-s^2 - (\epsilon + s^2)\sin(\phi)^2 \end{aligned} \qquad (1)$$

Barotropic ($\epsilon = 0$) Kelvin Waves

High zonal wavenumber Kelvin are equatorial modes even for $\epsilon = 0$

Barotropic Kelvin, ε=0, s=20

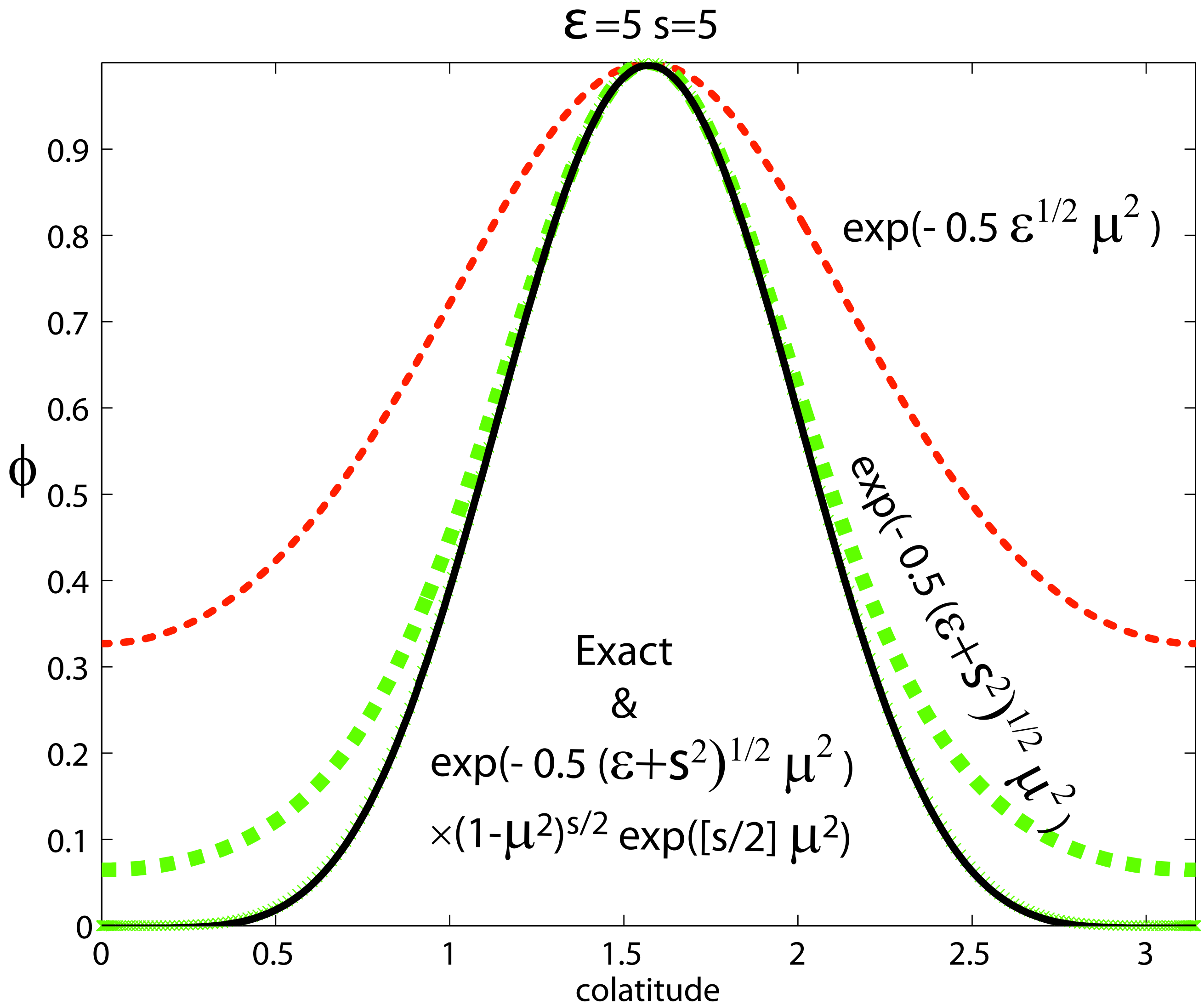


Boyd-Zhou Kelvin approximation is

$$\phi \approx (1-\mu^2)^{s/2}\exp((s/2)\mu^2)\times\exp(-(1/2)\sqrt{\epsilon + s^2}\mu^2)$$

$(\mu = \sin(\text{latitude}))$

Kelvin & approx. are solid black [graphically indistinguishable]



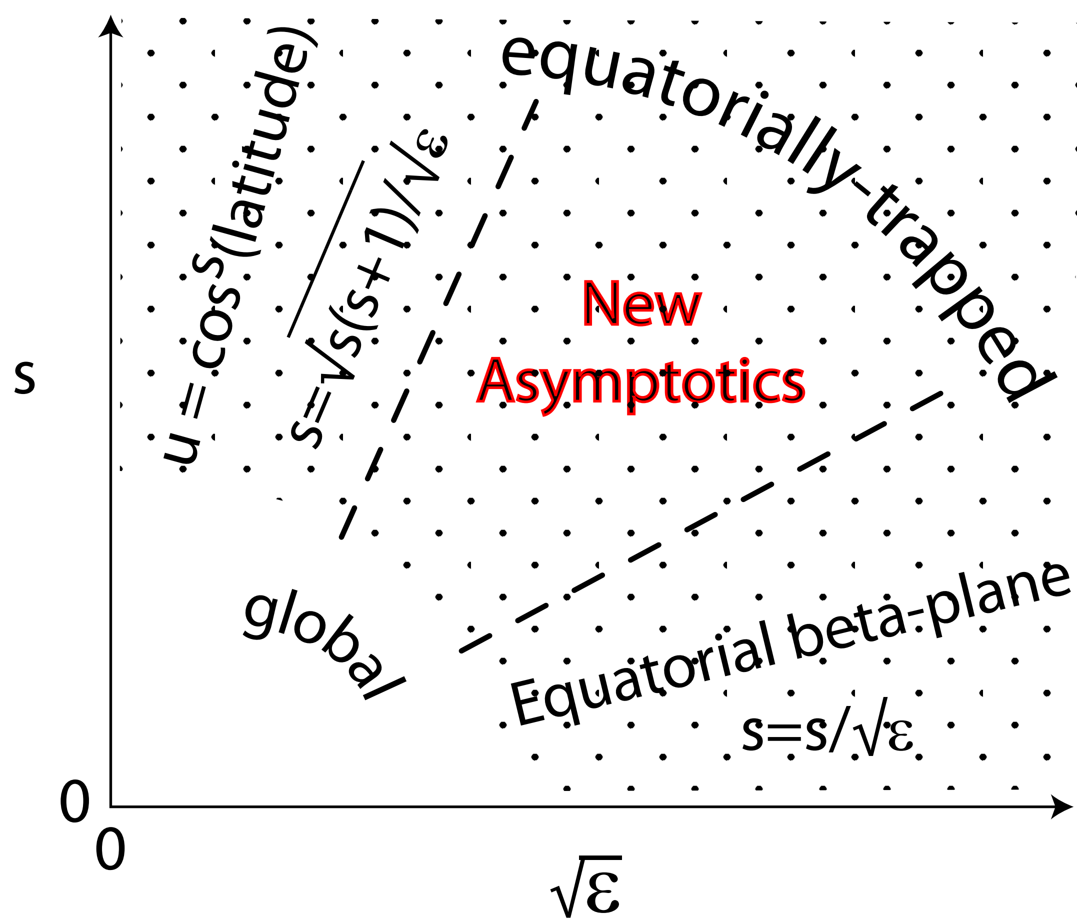
Uniform Validity

- New approximation is uniformly valid for

$$\sqrt{s^2 + \epsilon} >> 1$$

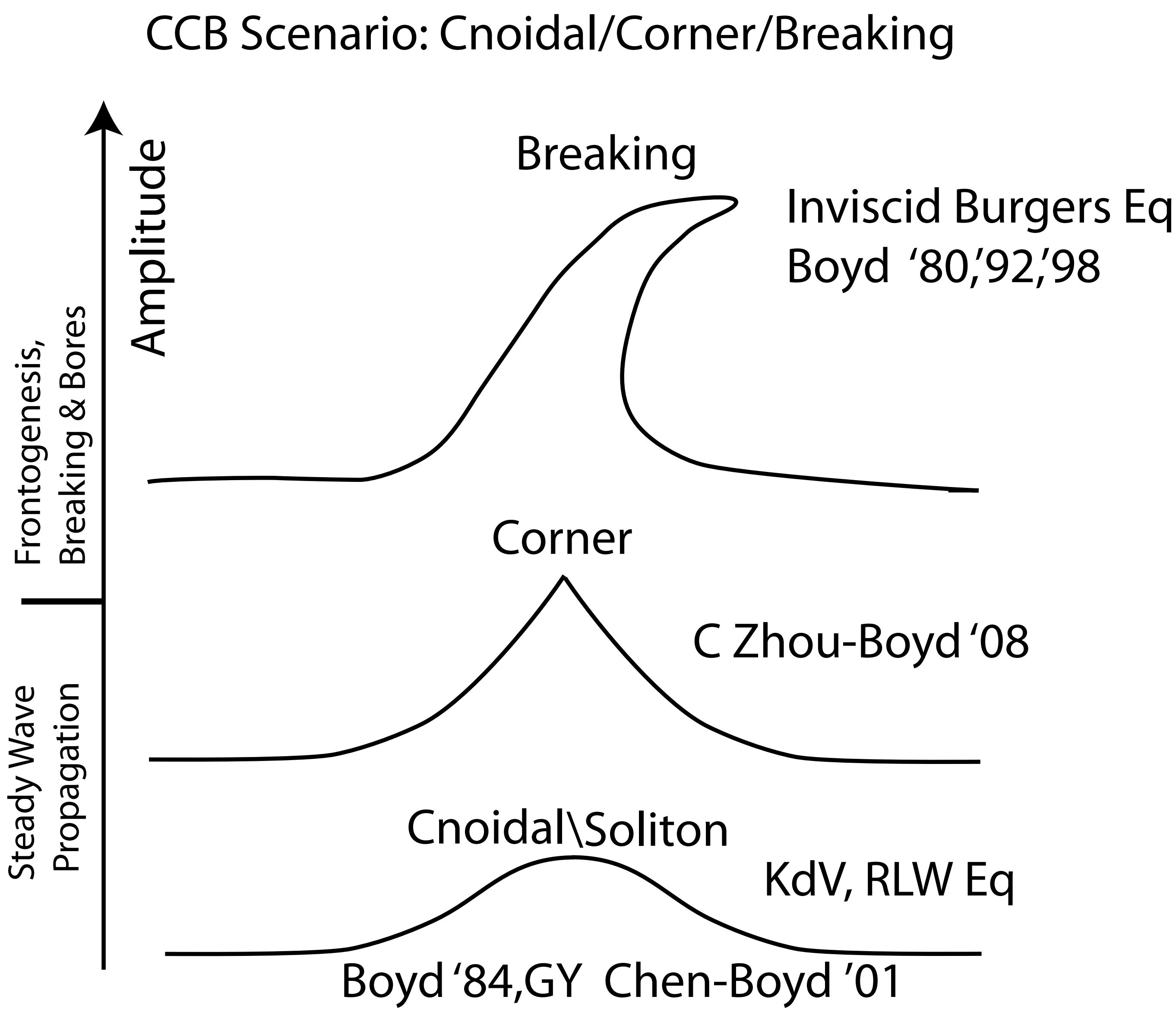
(shaded in figure)

- Though not strictly valid when both s and ϵ are $O(1)$, it is not a bad approximations



SECOND THEME: NONLINEAR KELVIN DYNAMICS

KELVIN MODE



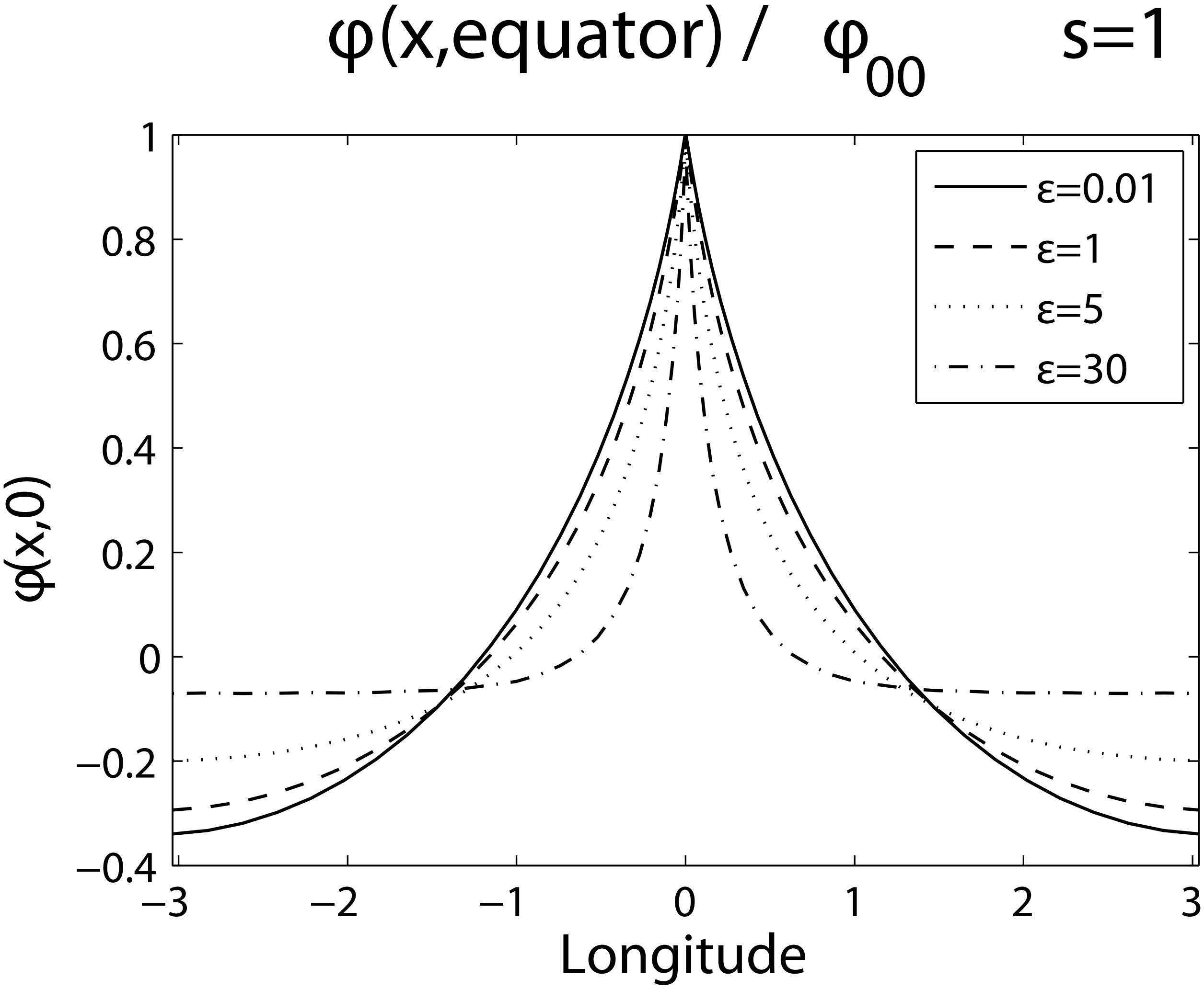
Definition 1 (Corner Wave) *A corner wave is a steadily traveling nonlinear wave in which the wave height function $u(x-ct)$ has a maximum which is a slope discontinuity.*

Table 1: Examples of Systems with Corner Waves and the CCB Scenario

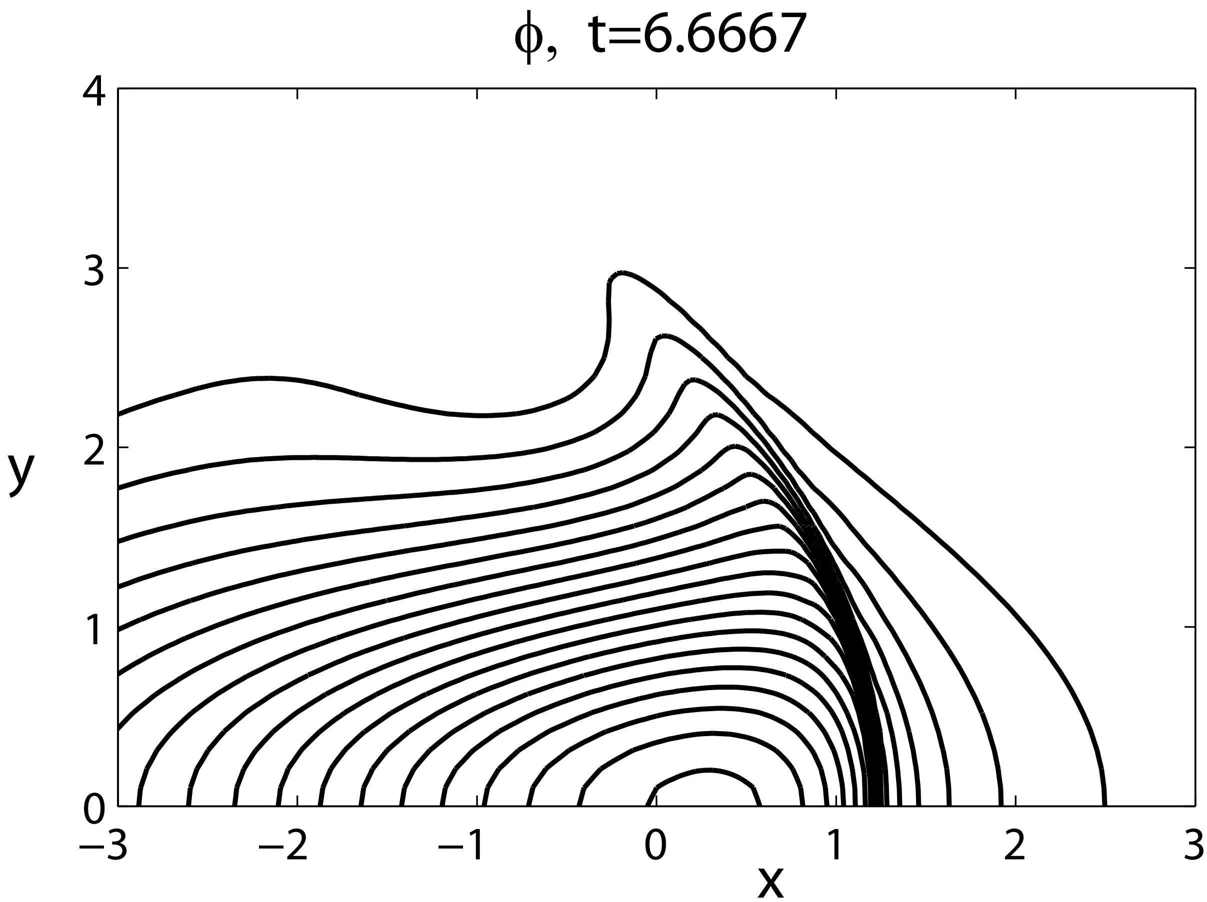
Eq. or Wave Name	Equations
Non-equatorial	
Equatorial Waves Barotropic Mode	$K_t + KK_x = \gamma \left\{ Y(t)e^{ix} + \overline{Y}(t)e^{-ix} \right\}$; $Y_t = -\gamma \hat{K}(x=1,t)$
Equatorial Waves Baroclinic Mode	3 coupled PDES in (x,t)
Resonant Triads, One Nondispersive	$u_t + uu_x = 2\mathrm{Re}\,(ikab\exp(-ikx))$; $a_t = -i\omega_a \overline{b\hat{u}_k}$; $b_t = -i\omega_b \overline{a\hat{u}_k}$
Equatorial Kelvin (4-mode Model)	4 coupled PDEs in x, t
Equatorial Kelvin (Shallow Water)	3 coupled PDEs in x, y, t (Shallow Water Eqs.)
Non-equatorial	
Surface Irrotational Water Waves	Euler equations in x, z
Boundary Waves on Vortex Patches	Two-space-dimensional Euler equations (x, y)
Camassa-Holm	$u_t - u_{xxt} + (2\kappa + 3u - 2u_{xx})u_x - uu_{xxx} = 0$
Ostrovsky-Hunter	$(u_t + uu_x)_x = u$
Gabov/ Shefter-Rosales	$(u_t + uu_x)_x = \int_0^{2\pi} \cos(x-y)\, u(y)\, dy$
Whitham	$(u_t + uu_x)_x = pb^2 \times \left\{ u - \int_0^{2\pi} \frac{b \cosh(b\{\lvert X-y\rvert - \pi\})}{2\sinh(b\pi)} u(y) dy \right\}$

Corner waves for different values of Lamb's paramter ϵ

CORNER WAVE is a POINT SINGULARITY
NOT a CREASE
NOT a CONE



Kelvin front CURVES because of resonance with gravity waves



HOUGH POINT CLOUDS INTO POLYNOMIALS

Hough' spherical harmonic Galerkin algorithm, with Longuet–Higgin's improvements, is very fast and spectrally accurate.

Mode classification is NOT a SLAM DUNK

Galerkin method generates POINT CLOUD: eigen-values at discrete ϵ.

Desired: CONTINUOUS BRANCHES

Other complications:

Kelvin mode $\Rightarrow$ GRAVITY WAVE as $\epsilon \to 0$

Yanai mode is "MIXED ROSSBY-GRAVITY"

Number of interior zeros may change with ϵ

Making Friends with Special Functions

CONCEPTUAL, QUALITATIVE:

Never-Out-of-Date Paradigms: Theorems, Asymptotics & Graphs

NUMERICAL:

Ancient Paradigm: Tables

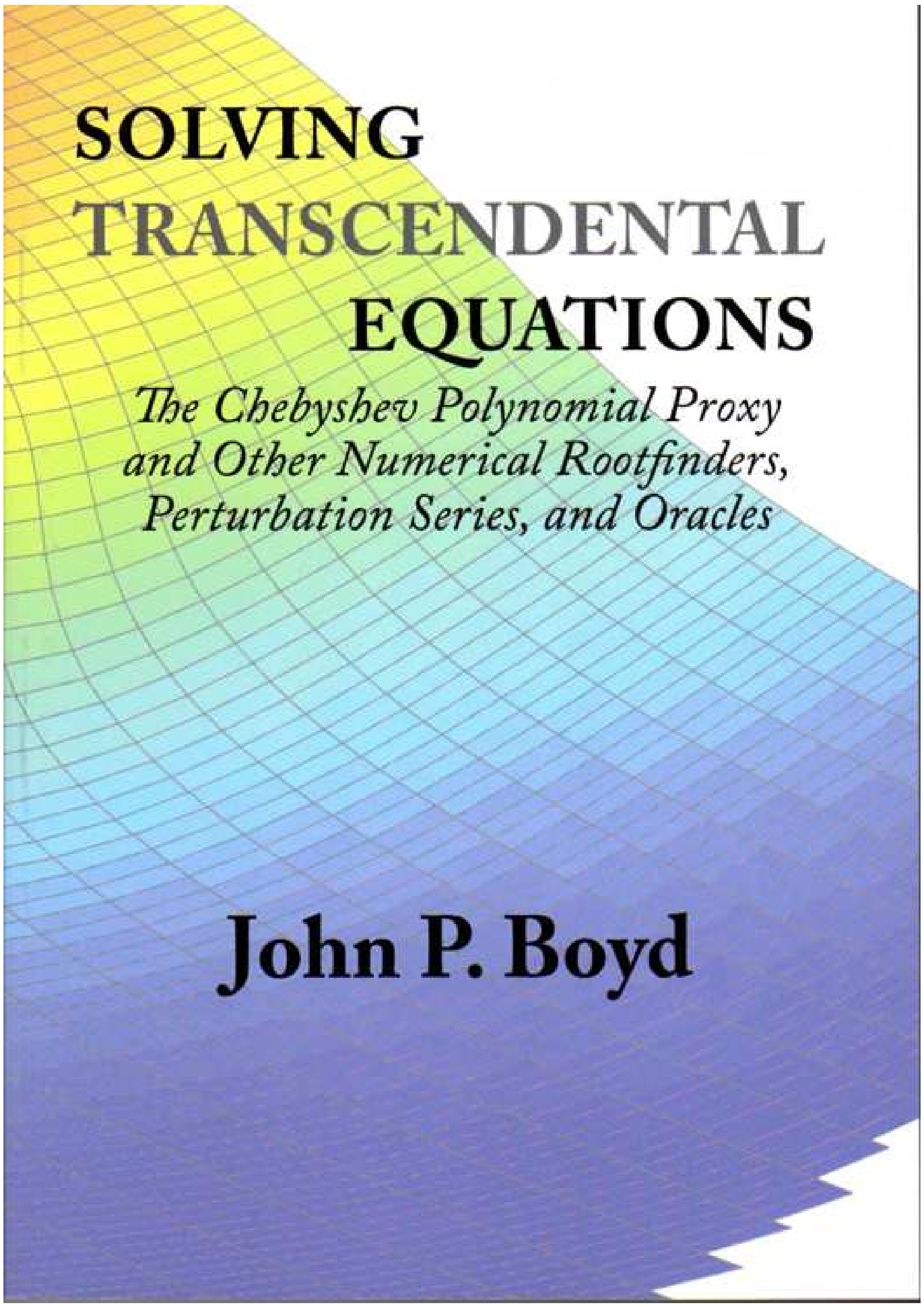
Newer Paradigm: Perturbation Series & Chebyshev Series

Emerging Paradigm: Matlab Code

Spherical harmonic Galerkin discretizations are tridiagonal (ϵ is eigenvalue) or otherwise very sparse. Power method allows very fast computation of a chosen mode for arbitrary parameter values without the need to compute all other modes if a Never-Failing-Initialization available.

Alternatives to Never-Failing-Initialization

Continuation, Davidenko Equation, etc., WORK but MANY POTENTIAL PROBLEMS Discussed in many references including:



Never-Failing-Initialization: Seven Series Options

small ε

large ε

spherical
harmonics

Hermite
functions

Pade from
small ε

Pade from
large ε

lall ε
TWO-POINT
PADE

Rational Chebyshev (TL series)
for each s

Double Rational Chebyshev

(ε, s)

Two-Point Padé Approximants

Example: Kelvin wave

The existence of such approximations suggests a unity of structure and identity in the Kelvin mode over all of $\epsilon \in [0, \infty]$.

Linear Polynomial/Linear Polynomial in $\sqrt{\epsilon}$ Matches (i) $\epsilon = 0$ limit (ii) two terms in $1/\sqrt{\epsilon}$:

$$c_{[1/1]}^{two-point} = \left(\sqrt{\frac{s+1}{s}} + 4\,\epsilon^{1/2} \sqrt{\frac{s+1}{s}} - 4\,\epsilon^{1/2} \right) \left(1 + 4\,\epsilon^{1/2} \sqrt{\frac{s+1}{s}} - 4\,\epsilon^{1/2} \right)^{-1} \qquad (2)$$

The next Kelvin approximation $c_{[2/2]}^{two-point}$ matches the first three terms of the large-ϵ expansion and two terms of the small-ϵ series [not shown]

The maximum relative error of the two-point Padé $c_{[2/2]}^{two-point}$ for Kelvin mode is only 0.0184 over all of $\epsilon \in [0, \infty]$.

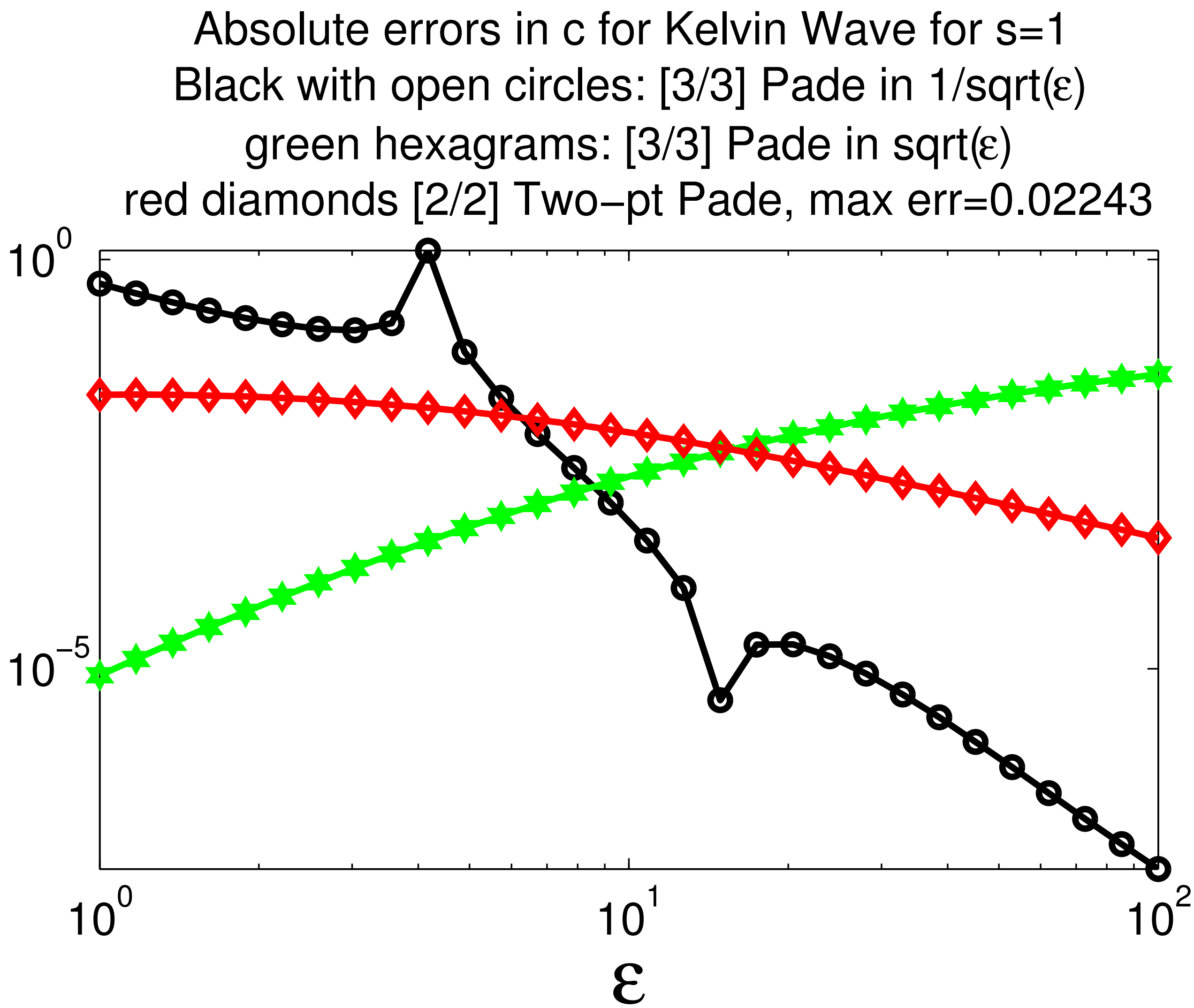


Figure 1: Errors in the small-ϵ and large-ϵ Padé $[3/3]$ approximations and also the quadratic-over-quadratic $c_{[2/2]}^{two-point}$ two-point rational approximation that for the phase speed for the Kelvin mode for $s = 1$.

Deriving Asymptotic Series by Galerkin Methods & Computer Algebra

- Galerkin Matrix Elements by Exact, Analytical Integration
  Hermite function basis [large ϵ]
  spherical harmonic basis [Small ϵ]
- Expand in ϵ or $1/\sqrt{\epsilon}$ & match powers
- Solve order-by-order in exact rational arithmetic

Low order small ϵ expansions by Dikii & Golitsyn and by Longuet-Higgins circa 1965

LH gave limited results for large ϵ

Exponential Smallness & Hermite Functions

[Define $\mu = \sin(latitude)$]

Key step in large ϵ, Hermite function asymptotics is

$$\boxed{\mu \to y/\sqrt{\sqrt{\epsilon}}}$$

Paradox: $y \in [-\epsilon^{-1/4}, \epsilon^{-1/4}]$

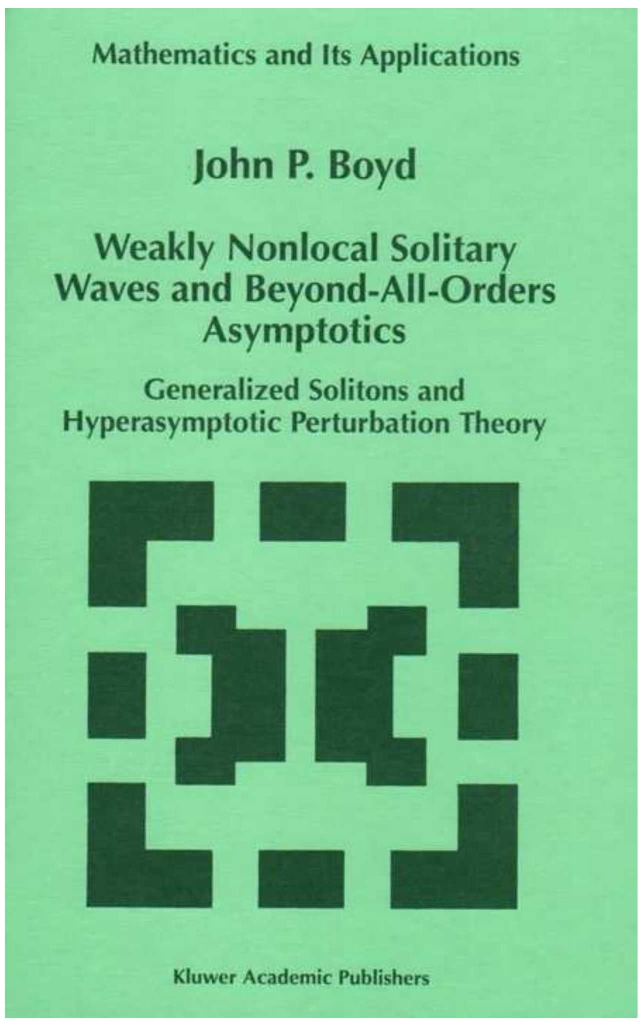
but Galerkin integrals are on $y \in [-\infty, \infty]$

- error is EXPONENTIALLY SMALL
- $\exp(-1/\sqrt{\epsilon})$ is INVISIBLE to ϵ-power series
- Coefficients of asymptotic series are EXACT & RATIONAL
- Series DIVERGE

Reviews on Exponential Smallness

"The Devil's Invention: Asymptotics, Superasymptotics and Hyperasymptotics", *Acta Applicandae*, **56**, 1-98 (1999).

" Hyperasymptotics and the Linear Boundary Layer Problem: Why Asymptotic Series Diverge, SIAM Rev. , 47, no. 3, 553-575 (2005)



SUMMARY

- Equatorial trapping depends on $s^2 + \epsilon$
  [zonal wavenumber (squared) plus Lamb's parameter]
- Kelvin Cnoidal Wave/Corner Wave/Breaking:

  Small amplitude Kelvin:
  cnoidal waves & solitons
  Largest non-breaking Kelvin wave is
  a corner wave
  Medium & large amplitude Kelvin:
  frontogenesis and breaking
- Hough point clouds can be connected by perturbation series and two-point Padé approximations

In preparation: “Hough Functions: Revisiting Longuet-Higgins' Masterwork Half a Century Later"