Linear and Weakly Nonlinear Energetics of Global Nonhydrostatic Normal Modes

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Introduction

- Conventional coarse resolution global circulation atmospheric models (AGCMs) neglect the vertical acceleration to prevent computational constraints related to stability of numerical solution in explicit numerical schemes (**hydrostatic assumption)** $\rightarrow$ **Hydrostatic AGCMs;**

- The increase of computer speed and memory, as well as the development of massively parallel computing techniques, has allowed the development of global nonhydrostatic modeling.

- Thus, with the increasing development of global nonhydrostatic AGCMs, it is important to understand the dynamics of these models in a theoretical point of view $\rightarrow$ For this purpose one needs to analyze the normal modes of global nonhydrostatic atmospheric models;

- Normal modes $\rightarrow$ small amplitude oscillations around a background state at rest and characterized by a stable stratification $\rightarrow$ eigensolutions of linearized PDEs;

References on Nonhydrostatic Normal Modes

➢ Linear theory of nonhydrostatic normal modes:

(i) Kasahara and Qian (MWR 2000) $\Rightarrow$ “Normal modes of a Global Nonhydrostatic Atmospheric Model”;

(ii) Qian and Kasahara (Pure Appl. Geophys., 2003) $\Rightarrow$ “Nonhydrostatic Atmospheric Normal Modes on Beta Planes”;

(iii) Kasahara (J. Meteor. Soc. Japan, 2003) $\Rightarrow$ “On the Nonhydrostatic Atmospheric Models with the Inclusion of the Horizontal Component of the Earth’s Angular Velocity” (non-traditional Coriolis terms);

(**iv) Kasahara (JAS 2003)** $\Rightarrow$ “The Roles of the Horizontal Component of the Earth’s Angular Velocity in Nonhydrostatic Linear Models”;

(**v) Kasahara (NCAR Report 2003)** $\Rightarrow$ “ Free Oscillations of Deep Nonhydrostatic Global Atmospheres: Theory and a Test of Numerical Schemes”;

Introduction

Goal of this Study

(i) First we further analyze the energetics of the linear eigenmodes of the shallow global nonhydrostatic model presented by Kasahara and Qian (2000);

(Ii) Then we extend the theory of global nonhydrostatic normal modes by accounting for the effect of nonlinearity.

Model and Governing Equations

- **Model**: shallow nonhydrostatic fluid over a rotating sphere of radius **a**;
  Traditional Approximation: r = **a** + z ≈ **a**, where **a** = 6370Km (Earth's radius) and z is the height above the earth's surface; $\partial / \partial r \approx \partial / \partial z$

- Governing Equations:

$$\frac{Du}{Dt} - \left(f + \frac{u \tan\varphi}{a} \right) \mathrm{v} = -\frac{1}{\rho a \cos\varphi} \frac{\partial p}{\partial \lambda} \qquad (1a)$$

$$\frac{D\mathrm{v}}{Dt} + \left(f + \frac{u \tan\varphi}{a} \right) \mathrm{u} = -\frac{1}{\rho a} \frac{\partial p}{\partial \varphi} \qquad (1b)$$

$$\delta_H \frac{D\mathrm{w}}{\mathrm{Dt}} = -g - \frac{1}{\rho} \frac{\partial p}{\partial z} \qquad (1c)$$

$$\frac{D\rho}{Dt} + \rho \left(\nabla \bullet \vec{V} + \frac{\partial \mathrm{w}}{\partial z} \right) = 0 \qquad (1d)$$

$$\frac{Dp}{Dt} = \gamma RT \frac{D\rho}{Dt} \qquad (1e)$$

$$p = \rho RT \qquad (1f)$$

$\lambda \rightarrow$ longitude;
$\varphi \rightarrow$ latitude;
$f = 2\Omega \sin\varphi \rightarrow$ Coriolis parameter;
$\gamma = C_p / C_v$;
g $\rightarrow$ gravity acceleration;
$\vec{V}$ = (u,v) $\rightarrow$ horizontal wind field;
p $\rightarrow$ pressure field;
T $\rightarrow$ Temperature field;
$\rho \rightarrow$ density field;
w $\rightarrow$ vertical component of wind field;

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{V} \bullet \nabla + \mathrm{w} \frac{\partial}{\partial z}; \qquad \vec{V} \bullet \nabla = \frac{u}{a\cos\varphi} \frac{\partial}{\partial \lambda} + \frac{\mathrm{v}}{\mathrm{a}} \frac{\partial}{\partial \varphi}; \qquad \nabla \bullet \vec{V} = \frac{1}{a \cos\varphi} \left[\frac{\partial u}{\partial \lambda} + \frac{\partial (\mathrm{v} \cos\varphi)}{\partial \varphi} \right]$$

Model and Governing Equations

- From the governing equations (1), we have considered small (but not infinitesimal) amplitude perturbations around a resting, hydrostatic and isothermal background state:

$u = u_0 + u'$; $v = v_0 + v'$; $w = w_0 + w'$, with $u_0 = v_0 = w_0 = 0$;
$p = p_0(z) + p'$; $\rho = \rho_0(z) + \rho'$; with $\mathbf{dp_0 / dz = -\rho_0 g}$ and $T = T_0 + T'$, with $\mathbf{T_0 = const}$; (2)

- Inserting (2) into (1) and retaning only the terms until second order in terms of perturbations:

$$\frac{\partial u'}{\partial t} - f\mathrm{v}' + \frac{1}{\rho_0 a\cos\varphi}\frac{\partial p'}{\partial\lambda} = -\left[\vec{V}'\bullet\nabla u' + w'\frac{\partial u'}{\partial z}\right] + \frac{u'\mathrm{v}'}{\mathrm{a}}\tan\varphi + \frac{\rho'}{\rho_0^{\,2} a\cos\varphi}\frac{\partial p'}{\partial\lambda} \quad \text{(3a)}$$

$$\frac{\partial \mathrm{v}'}{\partial t} + f\mathrm{u}' + \frac{1}{\rho_0 a}\frac{\partial p'}{\partial\varphi} = -\left[\vec{V}'\bullet\nabla \mathrm{v}' + \mathrm{w}'\frac{\partial \mathrm{v}'}{\partial z}\right] - \frac{u'^2}{a}\tan\varphi + \frac{\rho'}{\rho_0^{\,2} a}\frac{\partial p'}{\partial\varphi} \quad \text{(3b)}$$

$$\frac{\partial \mathrm{w}'}{\partial \mathrm{t}} + \frac{1}{\rho_0}\left(\frac{\partial p'}{\partial z} + \frac{g}{C_S^2}p' - \theta'\right) = -\left[\vec{V}'\bullet\nabla \mathrm{w}' + \mathrm{w}'\frac{\partial \mathrm{w}'}{\partial z}\right] + \frac{\rho'}{\rho_0^{\,2}}\frac{\partial p'}{\partial z} + g\left(\frac{\rho'}{\rho_0}\right)^2 \quad \text{(3c)}$$

$$\frac{1}{\rho_0 C_s^2}\left(\frac{\partial p'}{\partial t} - \rho_0 g\mathrm{w}'\right) + \nabla\bullet\vec{V}' + \frac{\partial \mathrm{w}'}{\partial z} = -\frac{1}{C_s^2}\left[\vec{V}'\bullet\nabla p' + \mathrm{w}'\frac{\partial p'}{\partial z}\right] - \rho'\left(\nabla\bullet\vec{V}' + \frac{\partial \mathrm{w}'}{\partial z}\right) + \frac{1}{C_s^2}\frac{T'}{T_0}\left[\frac{\partial p'}{\partial t} - \rho_0 g\mathrm{w}'\right] \quad \text{(3d)}$$

$$\frac{\partial\theta'}{\partial t} + \rho_0 N^2\mathrm{w}' = -\left[\vec{\mathrm{V}}'\bullet\nabla\theta' + \mathrm{w}'\frac{\partial\theta'}{\partial z}\right] + \frac{g}{C_S^2}\frac{T'}{T_0}\left(\frac{\partial p'}{\partial z} - \rho_0 g\mathrm{w}'\right) \quad \text{(3e)}$$

Model and Governing Equations

Where: $\theta' = \frac{g}{C_S^2} p' - g\rho'$ (3f) ; $\frac{p'}{p_0} = \frac{T'}{T_0} + \frac{\rho'}{\rho_0}$ (3g)

$$N^2 = -g\left(\frac{1}{\rho_0}\frac{d\rho_0}{dz} + \frac{g}{C_S^2}\right) = \frac{\kappa g}{H}$$

$$C_S^2 = \gamma R T_0 = \frac{gH}{1-\kappa}$$

$$H = \frac{RT_0}{g}$$

$$\kappa = \frac{R}{C_p}$$

- Following Kasahara and Qian (2000) we have rescaled the perturbations according to:

$$\begin{bmatrix} u' \\ \mathrm{v}' \\ \mathrm{w}' \\ \mathrm{p}' \\ \theta' \\ \rho' \end{bmatrix} = \begin{bmatrix} u\rho_0^{-\frac{1}{2}} \\ \mathrm{v}\rho_0^{-\frac{1}{2}} \\ \mathrm{w}\rho_0^{-\frac{1}{2}} \\ \mathrm{p}\rho_0^{\frac{1}{2}} \\ \theta\rho_0^{\frac{1}{2}} \\ \rho\rho_0^{\frac{1}{2}} \end{bmatrix} \quad (4)$$

Model and Governing Equations

- Substituting (4) into (3) we get:

$$\frac{\partial u}{\partial t} - f\mathrm{v} + \frac{1}{a\cos\varphi}\frac{\partial p}{\partial \lambda} = -\rho_0^{-\frac{1}{2}}\left\{\left[\vec{V}\bullet\nabla u + wL_z^{-}(u)\right] + \frac{u\mathrm{v}}{\mathrm{a}}\tan\varphi + \frac{\rho}{a\cos\varphi}\frac{\partial p'}{\partial \lambda}\right\} \quad \text{(5a)}$$

$$\frac{\partial \mathrm{v}}{\partial t} + f\mathrm{u} + \frac{1}{a}\frac{\partial p}{\partial \varphi} = -\rho_0^{-\frac{1}{2}}\left\{\left[\vec{V}\bullet\nabla \mathrm{v} + wL_z^{-}(\mathrm{v})\right] - \frac{u^2}{\mathrm{a}}\tan\varphi + \frac{\rho}{a}\frac{\partial p'}{\partial \varphi}\right\} \quad \text{(5b)}$$

$$\frac{\partial \mathrm{w}}{\partial \mathrm{t}} + \frac{\partial p}{\partial \mathrm{z}} + \Gamma p - \theta = -\rho_0^{-\frac{1}{2}}\left\{\left[\vec{V}\bullet\nabla \mathrm{w} + \mathrm{wL}_z^{-}(\mathrm{w})\right] + \rho\frac{\partial p}{\partial z} + g\rho^2\right\} \quad \text{(5c)}$$

$$\frac{1}{C_s^2}\frac{\partial p}{\partial t} + \nabla\bullet\vec{V} + \frac{\partial \mathrm{w}}{\partial \mathrm{z}} - \Gamma\mathrm{w} = -\rho_0^{-\frac{1}{2}}\left\{\frac{1}{C_s^2}\left[\vec{V}\bullet\nabla p + \mathrm{wL}_z^{+}(p)\right] - \rho\left(\nabla\bullet\vec{V} + L_Z^{-}(\mathrm{w})\right) + \frac{1}{C_s^2}\left(\frac{p}{RT_0} - \rho\right)\left(\frac{\partial p}{\partial t} - g\mathrm{w}\right)\right\} \quad \text{(5d)}$$

$$\frac{1}{N^2}\frac{\partial \theta}{\partial t} + \mathrm{w} = -\rho_0^{-\frac{1}{2}}\left\{\frac{1}{N^2}\left[\vec{\mathrm{V}}\bullet\nabla\theta + \mathrm{wL}_z^{+}(\theta)\right] + \frac{g}{N^2C_S^2}\left(\frac{\mathrm{p}}{\mathrm{RT}_0} - \rho\right)\left(\frac{\partial p}{\partial z} - g\mathrm{w}\right)\right\} \quad \text{(5e)}$$

Where,

$$L_z^{+}(\ ) = \frac{\partial}{\partial z} + \frac{1}{2\rho_0}\frac{d\rho_0}{dz} = \frac{\partial}{\partial z} - \frac{1}{2H} \qquad L_z^{-}(\ ) = \frac{\partial}{\partial z} - \frac{1}{2\rho_0}\frac{d\rho_0}{dz} = \frac{\partial}{\partial z} + \frac{1}{2H}$$

$$\Gamma = \frac{1}{2\rho_0}\frac{d\rho_0}{dz} + \frac{g}{C_S^2} = \frac{1-2\kappa}{2H}$$

Eigenmodes of the Linear Problem (Normal Modes)

➢ If the second-order nonlinear terms are disregarded, equations (5) become:

$$\frac{\partial u}{\partial t} - f\mathrm{v} + \frac{1}{a\cos\varphi}\frac{\partial p}{\partial \lambda} = 0 \qquad \text{(6a)}$$

$$\frac{\partial \mathrm{v}}{\partial t} + f\mathrm{u} + \frac{1}{a}\frac{\partial p}{\partial \varphi} = 0 \qquad \text{(6b)}$$

$$\frac{\partial \mathrm{w}}{\partial \mathrm{t}} + \frac{\partial p}{\partial z} + \Gamma p - \theta = 0 \qquad \text{(6c)}$$

$$\frac{1}{C_s^2}\frac{\partial p}{\partial t} + \nabla \bullet \vec{V} + \frac{\partial \mathrm{w}}{\partial z} - \Gamma \mathrm{w} = 0 \qquad \text{(6d)}$$

$$\frac{1}{N^2}\frac{\partial \theta}{\partial t} + \mathrm{w} = 0 \qquad \text{(6e)}$$

➢ The eigensolutions of (6) were determined by Kasahara and Qian (2000) for the following boundary conditions:

(i) w = 0 at z = 0 and at z = z_T (7a)
(ii) Periodic solutions in longitude (7b)
(iii) Regularity at the poles (7c)

Eigenmodes of the Linear Problem

The eigensolutions of (6) with boundary conditions (7) are given by:

$$\begin{bmatrix} u \\ \mathrm{v} \\ p \\ w \\ \theta \end{bmatrix} = \begin{bmatrix} U(\varphi)\xi(z) \\ iV(\varphi)\xi(z) \\ P(\varphi)\xi(z) \\ iP(\varphi)\eta(z) \\ P(\varphi)\Theta(z) \end{bmatrix} e^{is\lambda - i\sigma t} \qquad (8)$$

where:

$$\sigma\xi\left(\frac{1}{gH_e} - \frac{1}{C_s^2}\right) + \left(\frac{d\eta}{dz} - \Gamma\eta\right) = 0$$

$$-\sigma\Theta + N^2\eta = 0$$

$$\sigma\eta + \left(\frac{d\xi}{dz} + \Gamma\xi\right) - \Theta = 0$$

Vertical structure equations

$$-\sigma U - fV + \frac{sP}{a\cos\varphi} = 0$$

$$\sigma V + fU + \frac{1}{a}\frac{dP}{d\varphi} = 0$$

$$\frac{1}{a\cos\varphi}\left[sU + \frac{d}{d\varphi}(V\cos\varphi)\right] = \frac{\sigma P}{gH_e}$$

horizontal structure equations
(Laplace's tidal equations)

Eigenmodes of the Linear Problem

- The vertical structure equations can be written in terms of η as follows:

$$\frac{d^2\eta}{dz^2} + \left(\lambda - \Gamma^2\right)\eta = 0 \quad ; \qquad \text{with BCs: } \eta = 0 \text{ at } z = 0 \text{ and at } z = z_T \;;$$

$$\lambda = \left(\frac{1}{gH_e} - \frac{1}{C_s^2}\right)\left(N^2 - \delta_H \sigma^2\right)$$

- The solution is given by: $\eta(z) = A_k \sin\left(\frac{k\pi}{z_T} z\right)$, k = 1, 2, 3 ... , provided that

$$\lambda_k = \left(\frac{k\pi}{z_T}\right)^2 + \Gamma^2$$

Eigenvalues of the vertical structure equations

- Separation constant: $H_e = \frac{C_s^2}{g}\left(1 + \frac{\lambda_k C_s^2}{N^2 - \delta_H \sigma^2}\right)^{-1}$

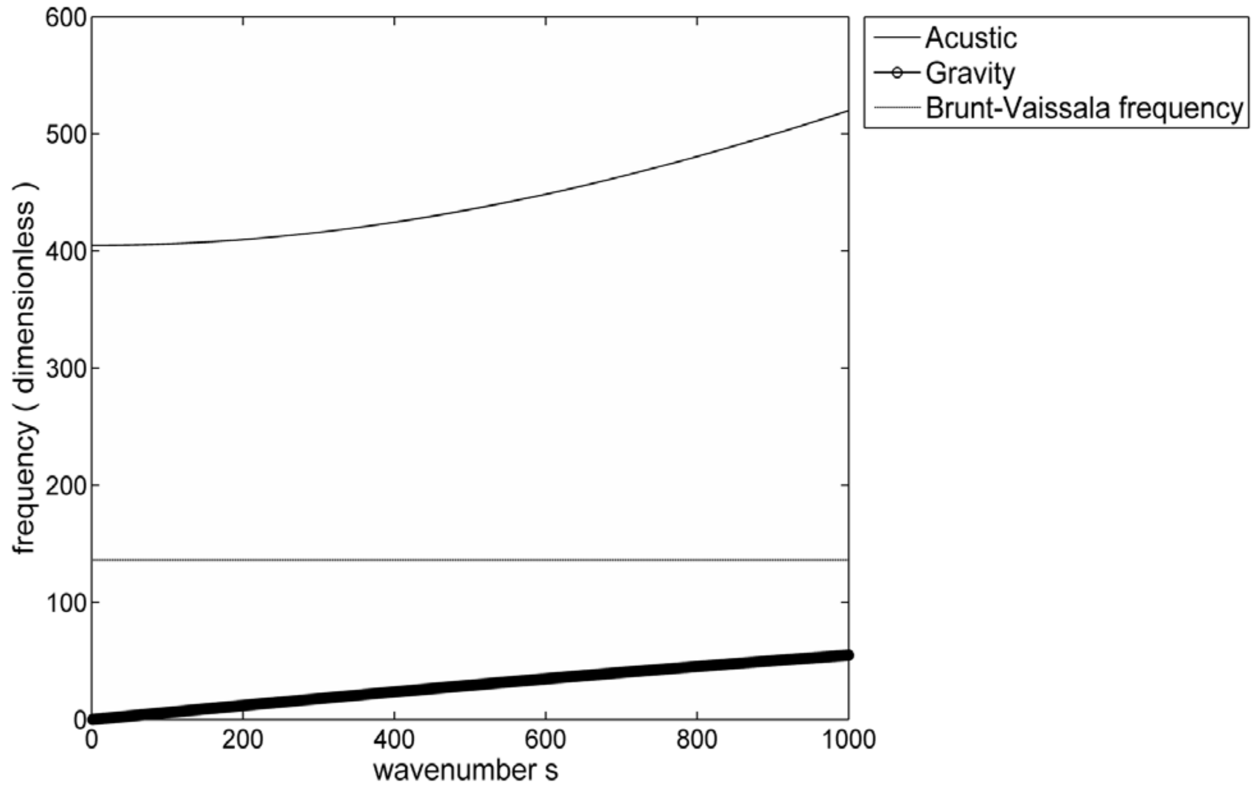
$$\sigma = F\left(s, l, H_e\right)$$

→ Eigenvalues of the Laplace's tidal equations

Eigenmodes of the Linear Problem

- Different oscillation regimes: (I) $H_e > \frac{C_S^2}{g} \rightarrow \sigma^2 > N^2 \rightarrow$ inertio-acoustic modes;

  (ii) $H_e < \frac{C_S^2}{g} \rightarrow \sigma^2 < N^2 \rightarrow$ inertio-gravity modes;



Dispersion curves for acoustic and gravity modes for l = 0, k = 1, and symmetric about the equator.

Eigenmodes of the Linear Problem

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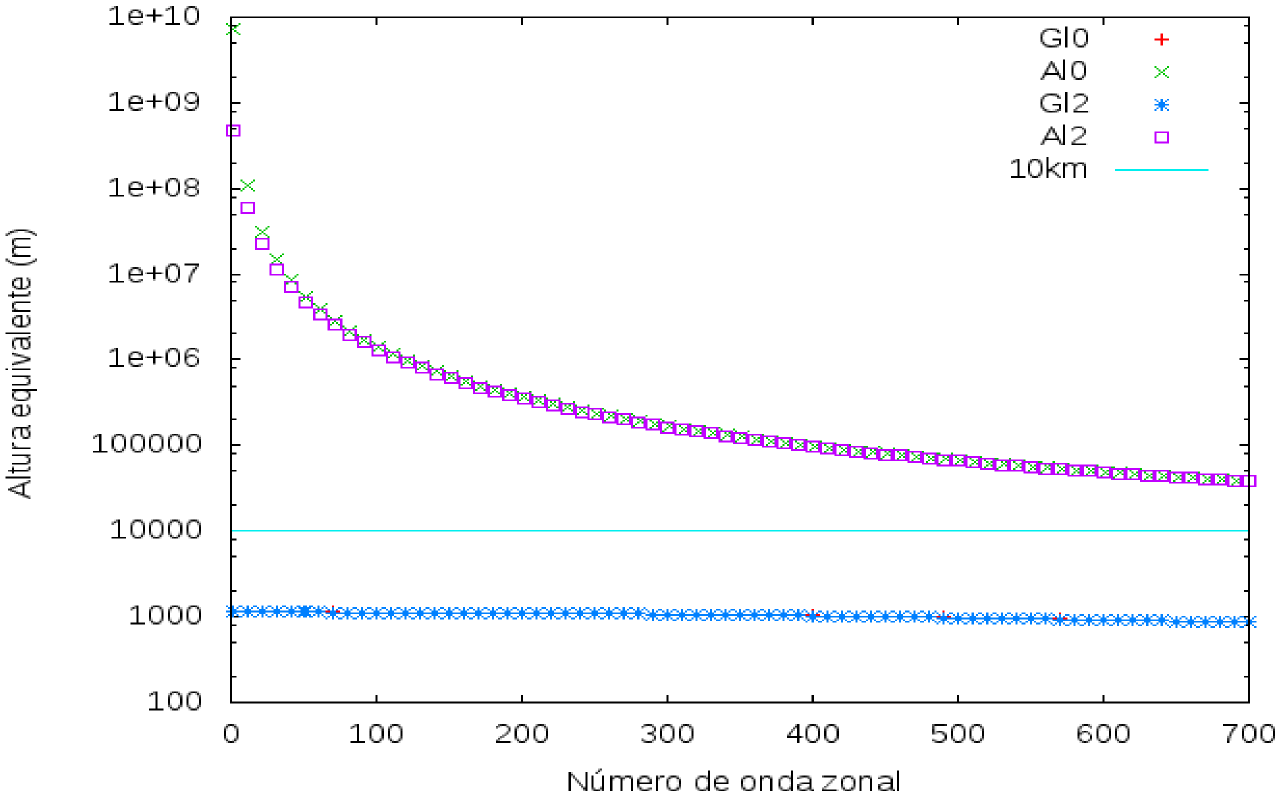


Equivalent heights H_e for acoustic and gravity modes for l = 0, k = 1, and symmetric about the equator.

Energetics of Normal modes

- Kasahara and Qian (2000) have demonstrated the orthogonality condition for the eigenmodes:

$$(i\sigma_j - i\sigma_k)\int_0^{z_T}\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\left(u_j u_k^* + \mathrm{v}_j \mathrm{v}_k^* + \delta_H \mathrm{w}_j \mathrm{w}_k^*\right) + \frac{\mathrm{p}_j \mathrm{p}_k^*}{\mathrm{C}_S^2} + \frac{\theta_j \theta_k^*}{\mathrm{N}^2}\right] a^2 \cos\varphi\, d\varphi\, d\lambda\, dz = 0$$

- For the case j = k we have the total energy of the j-th eigenmode of the system:

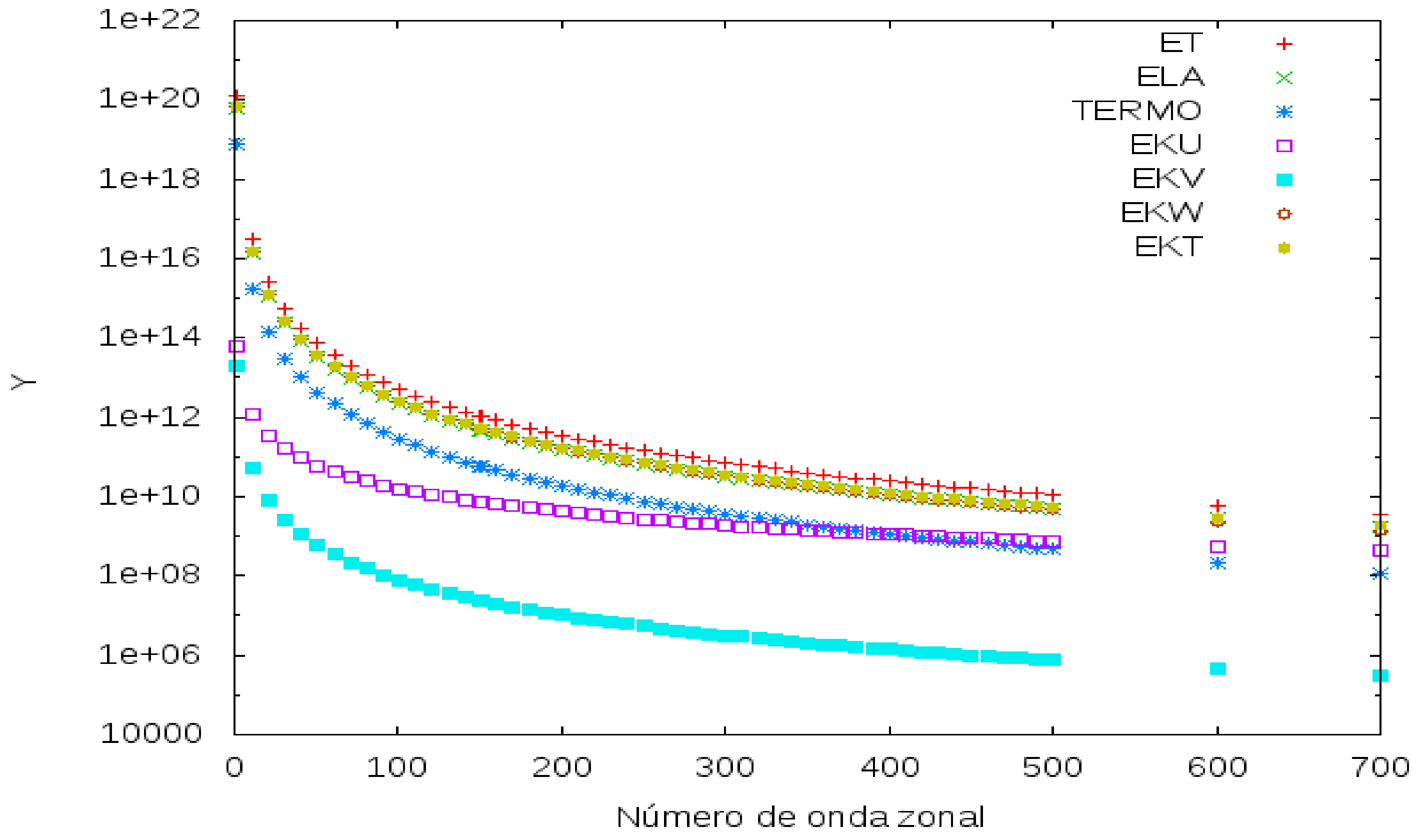
$$ET_j = \int_0^{z_T}\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[K_j + E_j + A_j\right] a^2 \cos\varphi\, d\varphi\, d\lambda\, dz > 0$$

Where: $K_j = \frac{1}{2}\int_0^{z_T}\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\left(u_j^{\,2} + \mathrm{v}_j^{\,2} + \delta_H \mathrm{w}_j^{\,2}\right)\right] a^2 \cos\varphi\, d\varphi\, d\lambda\, dz \longrightarrow$ Kinetic energy

$E_j = \frac{1}{2}\int_0^{z_T}\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{p}_j^{\,2}}{\mathrm{C}_S^2} a^2 \cos\varphi\, d\varphi\, d\lambda\, dz \longrightarrow$ elastic energy

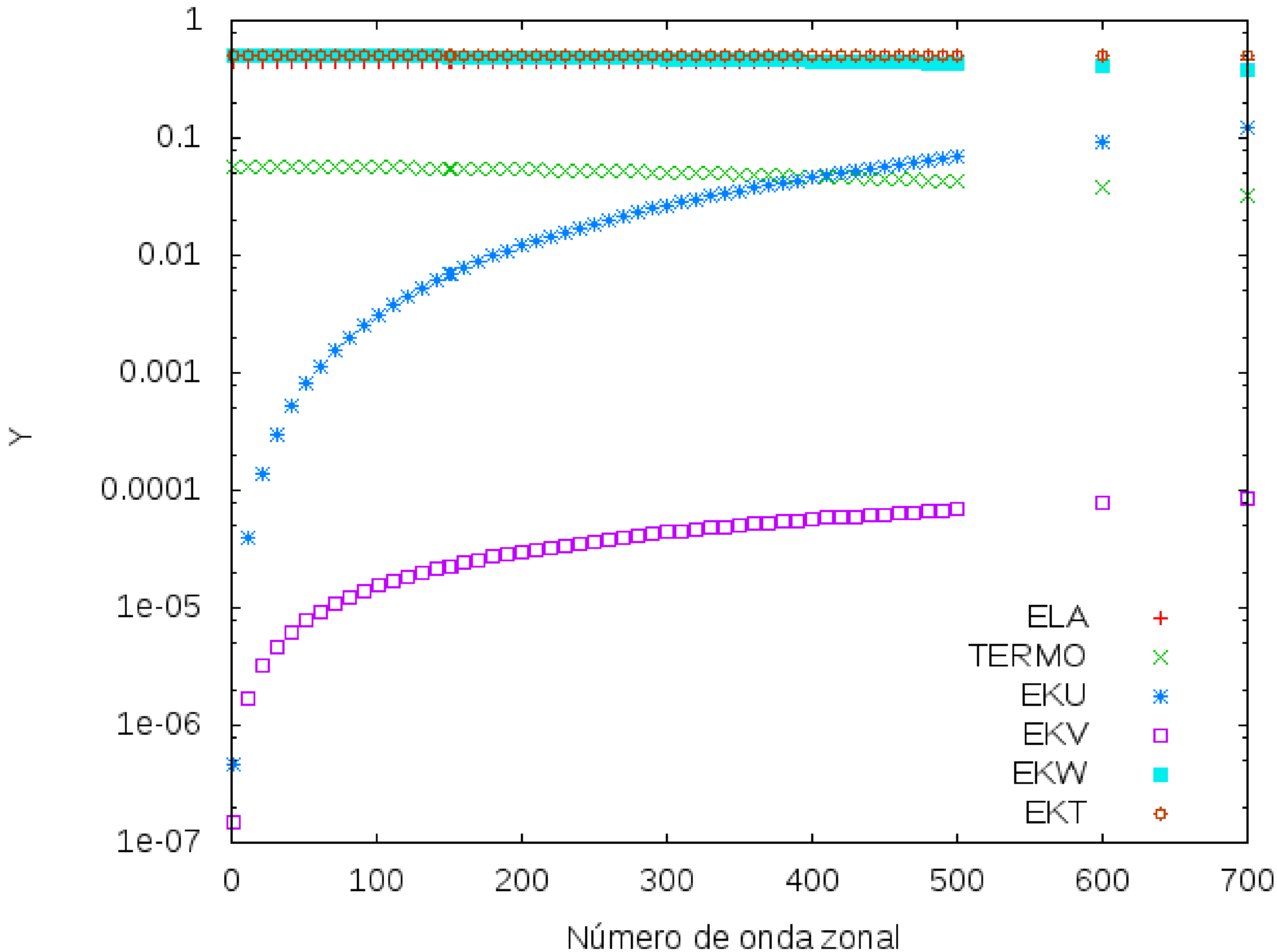
$A_j = \frac{1}{2}\int_0^{z_T}\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\theta_j^{\,2}}{\mathrm{N}^2} a^2 \cos\varphi\, d\varphi\, d\lambda\, dz \longrightarrow$ Termobaric energy

Energetics of Normal modes



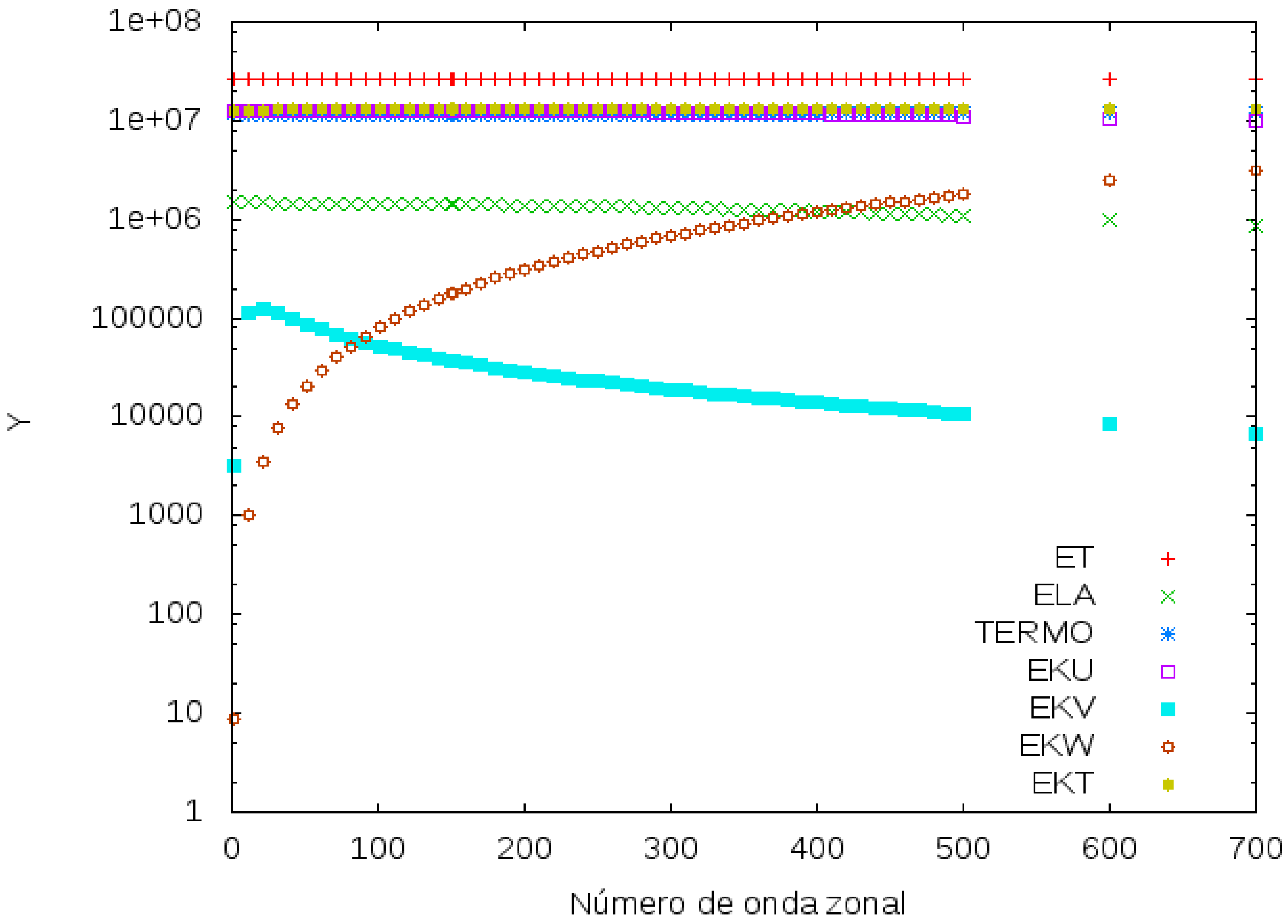
Energetics of eastward inertio-acoustic modes with meridional index l = 0 and k = 1.

Energetics of Normal modes



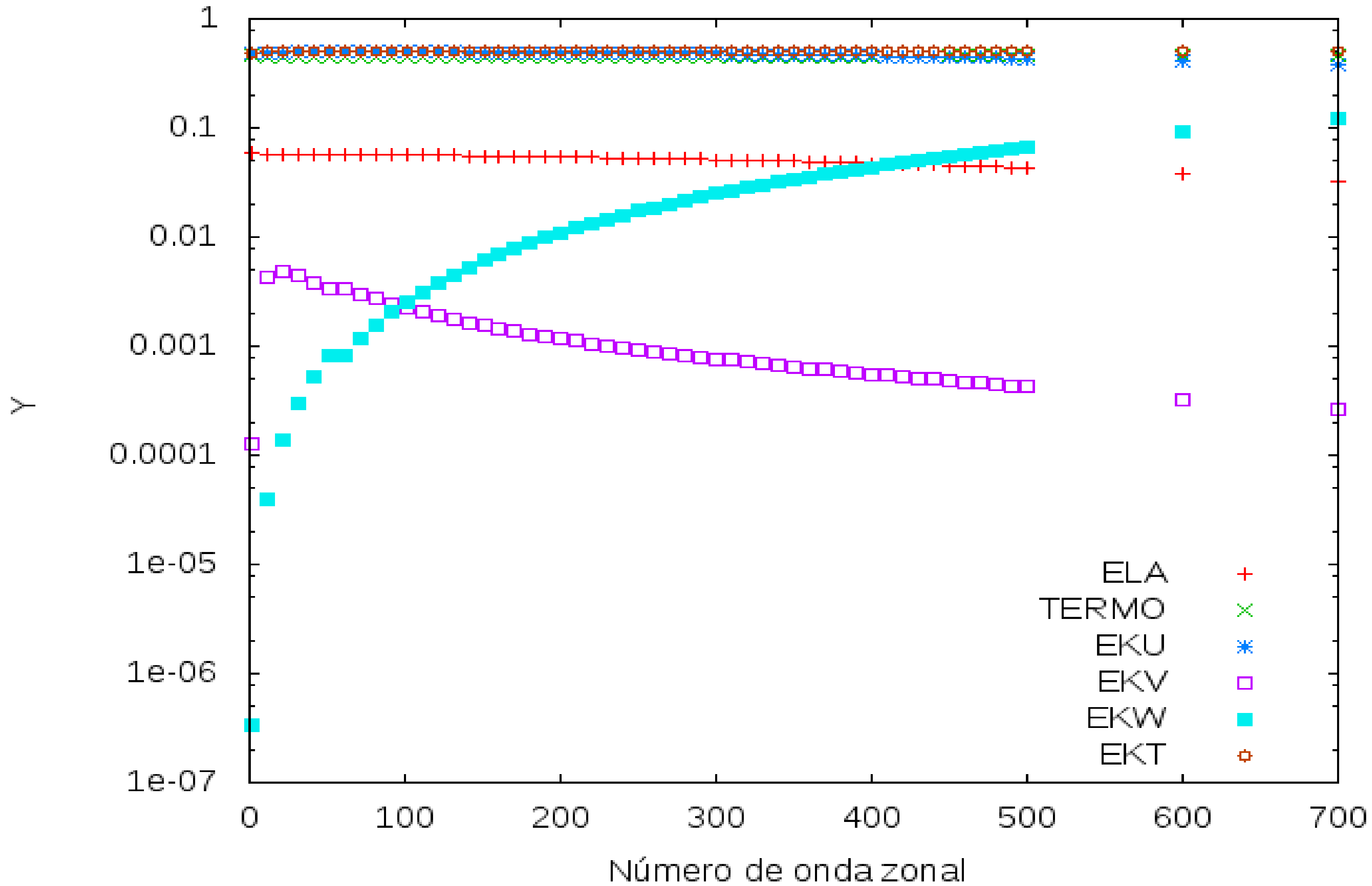
Energetics of eastward inertio-acoustic modes with meridional index l = 0 and k = 1. Each energy type is normalized by total energy.

Energetics of Normal modes



Energetics of eastward inertio-gravity modes with meridional index l = 0 and k = 1.

Energetics of Normal modes



Energetics of eastward inertio-gravity modes with meridional index l = 0 and k = 1. Each energy type is normalized by total energy.

Resonant Nonlinear Interactions of Global Nonhydrostatic Modes

$$\frac{\partial u}{\partial t} - f\mathrm{v} + \frac{1}{a\cos\varphi}\frac{\partial p}{\partial \lambda} = -\rho_0^{-\frac{1}{2}}\left\{\left[\vec{V}\bullet\nabla u + wL_z^-(u)\right] + \frac{u\mathrm{v}}{\mathrm{a}}\tan\varphi + \frac{\rho}{a\cos\varphi}\frac{\partial p'}{\partial \lambda}\right\} \quad (5a)$$

$$\frac{\partial \mathrm{v}}{\partial t} + f\mathrm{u} + \frac{1}{a}\frac{\partial p}{\partial \varphi} = -\rho_0^{-\frac{1}{2}}\left\{\left[\vec{V}\bullet\nabla \mathrm{v} + wL_z^-(\mathrm{v})\right] - \frac{u^2}{\mathrm{a}}\tan\varphi + \frac{\rho}{a}\frac{\partial p'}{\partial \varphi}\right\} \quad (5b)$$

$$\frac{\partial \mathrm{w}}{\partial \mathrm{t}} + \frac{\partial p}{\partial \mathrm{z}} + \Gamma p - \theta = -\rho_0^{-\frac{1}{2}}\left\{\left[\vec{V}\bullet\nabla \mathrm{w} + \mathrm{w}\mathrm{L}_z^-(\mathrm{w})\right] + \rho\frac{\partial p}{\partial z} + g\rho^2\right\} \quad (5c)$$

$$\frac{1}{C_s^2}\frac{\partial p}{\partial t} + \nabla\bullet\vec{V} + \frac{\partial \mathrm{w}}{\partial z} - \Gamma\mathrm{w} = \rho_0^{-\frac{1}{2}}\left\{-\frac{1}{C_S^2}\left[\vec{V}\bullet\nabla p + \mathrm{w}\mathrm{L}_z^+(\mathrm{p})\right] - \rho\left(\nabla\bullet\vec{V} + \mathrm{L}_z^-(\mathrm{w})\right) + \frac{1}{C_S^2}\left(\frac{p}{RT_0} - \rho\right)\left(\frac{\partial p}{\partial t} - g\mathrm{w}\right)\right\} \quad (5d)$$

$$\frac{1}{N^2}\frac{\partial \theta}{\partial t} + \mathrm{w} = \rho_0^{-\frac{1}{2}}\left\{-\frac{1}{N^2}\left[\vec{V}\bullet\nabla\theta + \mathrm{w}\mathrm{L}_z^+(\theta)\right] + \frac{g}{N^2C_S^2}\left(\frac{p}{RT_0} - \rho\right)\left(\frac{\partial p}{\partial t} - g\mathrm{w}\right)\right\} \quad (5e)$$

Where,

$$L_z^+(\;) = \frac{\partial}{\partial z} + \frac{1}{2\rho_0}\frac{d\rho_0}{dz} = \frac{\partial}{\partial z} - \frac{1}{2H} \qquad L_z^-(\;) = \frac{\partial}{\partial z} - \frac{1}{2\rho_0}\frac{d\rho_0}{dz} = \frac{\partial}{\partial z} + \frac{1}{2H}$$

$$\Gamma = \frac{1}{2\rho_0}\frac{d\rho_0}{dz} + \frac{g}{C_S^2} = \frac{1-2\kappa}{2H}$$

Resonant Interactions of nonhydrostaic nonrmal modes: general case

Ansatz $\Rightarrow$ Solution with **three** modes:

$$\begin{bmatrix} u \\ v \\ w \\ p \\ \theta \\ \rho \end{bmatrix}(\lambda,\varphi,z,t) = A_1(t)\begin{bmatrix} U_1(\varphi,z) \\ iV_1(\varphi,z) \\ iW_1(\varphi,z) \\ P_1(\varphi,z) \\ \theta_1(\varphi,z) \\ \rho_1(\varphi,z) \end{bmatrix} e^{is_1\lambda - i\sigma_1 t} + A_2(t)\begin{bmatrix} U_2(\varphi,z) \\ iV_2(\varphi,z) \\ iW_2(\varphi,z) \\ P_2(\varphi,z) \\ \theta_2(\varphi,z) \\ \rho_2(\varphi,z) \end{bmatrix} e^{is_2\lambda - i\sigma_2 t} + A_3(t)\begin{bmatrix} U_3(\varphi,z) \\ iV_3(\varphi,z) \\ iW_3(\varphi,z) \\ P_3(\varphi,z) \\ \theta_3(\varphi,z) \\ \rho_3(\varphi,z) \end{bmatrix} e^{is_3\lambda - i\sigma_3 t} + C.C$$

With the following resonance relations satisfied:

$$I_z = \int_0^{z_T} \rho_0^{-\frac{1}{2}} \cos\left[(k_1 \pm k_2 \pm k_3)\frac{\pi z}{z_T}\right] dz$$

$k_3 = k_1 + k_2$ (not excluding)

$s_1 = s_2 + s_3$

$\sigma_1 = \sigma_2 + \sigma_3$

condition for meridional structures satisfied

→ **Nonlinear resonant triad interaction conditions**

Resonant Interactions between Acoustic and Gravity Modes

- Substituting the ansatz into the PDEs (5) we get:

$$ET_1 \frac{dA_1}{dt} = i\alpha_1^{23} A_2 A_3$$

$$ET_2 \frac{dA_2}{dt} = i\alpha_2^{13} A_1 A_3^*$$

$$ET_3 \frac{dA_3}{dt} = i\alpha_3^{12} A_1 A_2^*$$

α_1^{23}, α_2^{13} , α_3^{12} $\Rightarrow$ Nonlinear coupling constants;

Resonant Interactions of nonhydrostaic nonrmal modes: general case

$$\alpha_1^{23} = \int_0^{z_T} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[N_u^{(2,3)} u_1^* + N_\mathrm{v}^{(2,3)} \mathrm{v}_1^* + N_\mathrm{w}^{(2,3)} \mathrm{w}_1^* + N_u^{(2,3)} u_1^* + N_\mathrm{p}^{(2,3)} \mathrm{p}_1^* + N_\theta^{(2,3)} \theta_1^* \right] a^2 \cos\varphi d\varphi \rho_0^{-\frac{1}{2}} dz$$

$$\alpha_2^{13} = \int_0^{z_T} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[N_u^{(1,3)} u_2^* + N_\mathrm{v}^{(1,3)} \mathrm{v}_2^* + N_\mathrm{w}^{(1,3)} \mathrm{w}_2^* + N_u^{(1,3)} u_2^* + N_\mathrm{p}^{(1,3)} \mathrm{p}_2^* + N_\theta^{(1,3)} \theta_2^* \right] a^2 \cos\varphi d\varphi \rho_0^{-\frac{1}{2}} dz$$

$$\alpha_3^{12} = \int_0^{z_T} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[N_u^{(1,2)} u_3^* + N_\mathrm{v}^{(1,2)} \mathrm{v}_3^* + N_\mathrm{w}^{(1,2)} \mathrm{w}_3^* + N_u^{(1,2)} u_3^* + N_\mathrm{p}^{(1,2)} \mathrm{p}_3^* + N_\theta^{(1,2)} \theta_3^* \right] a^2 \cos\varphi d\varphi \rho_0^{-\frac{1}{2}} dz$$

Resonant Interactions of nonhydrostaic nonrmal modes: general case

$$N_u^{(2,3)} = -\left[u_2 \frac{is_3 u_3}{a\cos\varphi} + \frac{\mathrm{v}_2}{a}\frac{\partial u_3}{\partial\varphi} + \mathrm{w}_2 \mathrm{L}_z^{-}(u_3)\right] + \frac{u_2 \mathrm{v}_3}{\mathrm{a}}\tan\varphi + \frac{\rho_2}{a\cos\varphi} is_3 p_3 + CP$$

$$N_{\mathrm{v}}^{(2,3)} = -\left[u_2 \frac{is_3 \mathrm{v}_3}{a\cos\varphi} + \frac{\mathrm{v}_2}{a}\frac{\partial \mathrm{v}_3}{\partial\varphi} + \mathrm{w}_2 \mathrm{L}_z^{-}(\mathrm{v}_3)\right] - \frac{u_2 u_3}{\mathrm{a}}\tan\varphi + \frac{\rho_2}{a}\frac{\partial p_3}{\partial\varphi} + CP$$

$$N_{\mathrm{w}}^{(2,3)} = -\delta_H\left[\frac{\mathrm{u}_2}{\mathrm{a}\cos\varphi}\mathrm{is}_3\mathrm{w}_3 + \frac{\mathrm{v}_2}{a}\frac{\partial \mathrm{w}_3}{\partial\varphi} + \mathrm{w}_2 L_z^{-}(\mathrm{w}_3)\right] + \rho_2 L_z^{+}(p_3) + g\rho_2\rho_3 + CP$$

$$N_p^{(2,3)} = -\frac{1}{C_S^2}\left[\frac{u_2}{a\cos\varphi} is_3 p_3 + \frac{\mathrm{v}_2}{a}\frac{\partial \mathrm{p}_3}{\partial\varphi} + \mathrm{w}_2 \mathrm{L}_z^{+}(\mathrm{p}_3)\right] - \rho_2\left[\frac{1}{a\cos\varphi}\left(is_3 u_3 + \frac{\partial(\mathrm{v}_3\cos\varphi)}{\partial\varphi}\right) + L_z^{-}(\mathrm{w}_3)\right]$$
$$+\frac{1}{C_S^2}\left(\frac{p_2}{RT_0} - \rho_2\right)(-i\sigma_3 p_3 - g\mathrm{w}_3) + CP$$

$$N_\theta^{(2,3)} = -\frac{1}{N^2}\left[\frac{u_2}{a\cos\varphi} is_3\theta_3 + \frac{\mathrm{v}_2}{a}\frac{\partial\theta_3}{\partial\varphi} + \mathrm{w}_2 \mathrm{L}_z^{+}(\theta_3)\right] + \frac{g}{C_S^2 N^2}\left(\frac{p_2}{RT_0} - \rho_2\right)(-i\sigma_3 p_3 - g\mathrm{w}_3) + CP$$

Resonant Interactions of nonhydrostaic nonrmal modes

- **From the complex amplitude equations it is easy to get the energy equations:**

$$ET_1 \frac{d|A_1|^2}{dt} = -\alpha_1^{23} \operatorname{Im}\left(A_1 A_2^* A_3^*\right)$$

$$ET_2 \frac{d|A_2|^2}{dt} = \alpha_2^{13} \operatorname{Im}\left(A_1 A_2^* A_3^*\right)$$

$$ET_3 \frac{d|A_3|^2}{dt} = \alpha_3^{12} \operatorname{Im}\left(A_1 A_2^* A_3^*\right)$$

- **Condition for total energy to be conserved within a resonant triad interaction is:**

$$-\alpha_1^{23} + \alpha_2^{13} + \alpha_3^{12} = 0$$

Mode 1 $\Rightarrow$ unstable mode of the triad.

Resonant Interactions between Acoustic and Gravity Modes

➢ Analytical solutions of the conservative triad equations, assuming that

($\left|\alpha_3^{12}\right| < \left|\alpha_2^{13}\right| < \left|\alpha_1^{23}\right|$) and the amplitude of mode 1 is zero initially:

$$ET_1\left|A_1(t)\right|^2 = \left|A_2(0)\right|^2\left(\left|\frac{\alpha_1^{23}}{\alpha_2^{13}}\right|\right)sn^2\left(\frac{u}{m}\right)$$

$$ET_2\left|A_2(t)\right|^2 = \left|A_2(0)\right|^2 cn^2\left(\frac{u}{m}\right)$$

$$ET_3\left|A_3(t)\right|^2 = \left|A_3(0)\right|^2 dn^2\left(\frac{u}{m}\right)$$

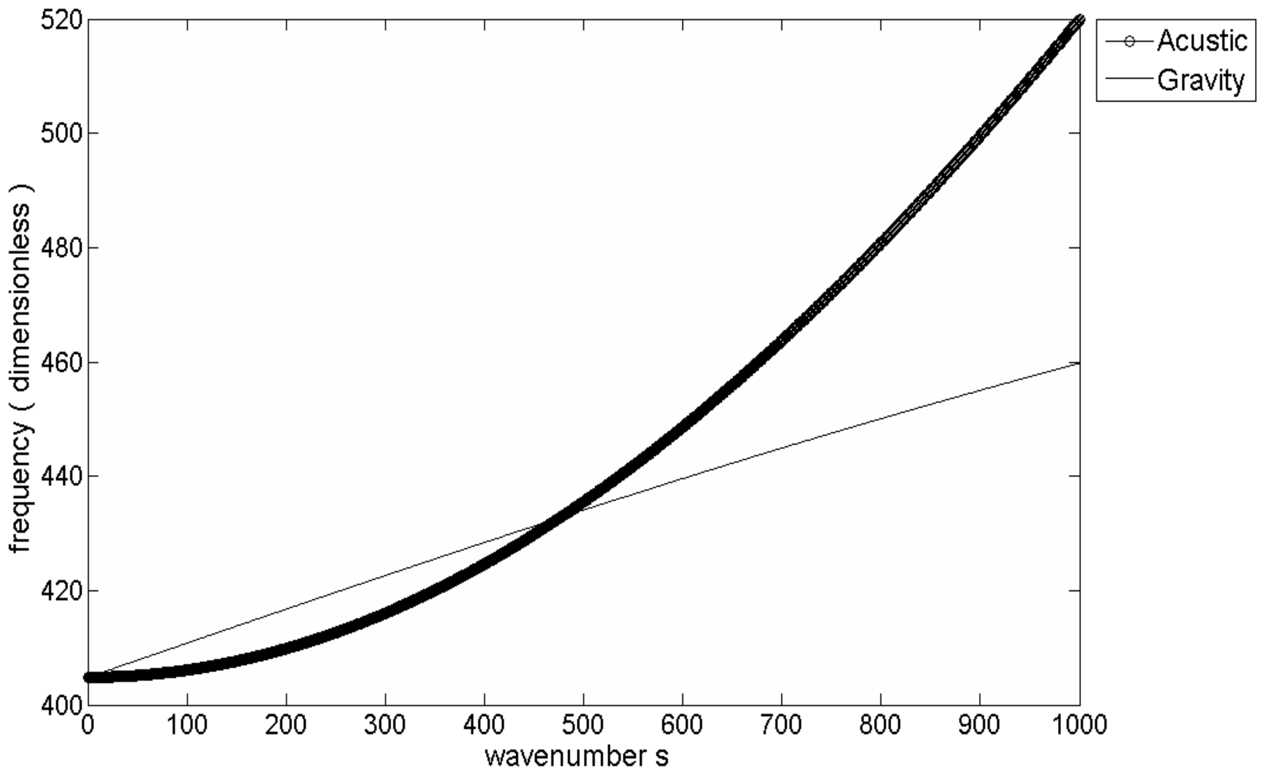
Where sn, cn and dn are the Jacobian Elliptic functions, with argument u and parameter m given by

$$u = \left|A_3(0)\right|\left|\alpha_1^{23}\alpha_2^{13}\right|t$$

$$m = \frac{\alpha_3^{12}}{\alpha_2^{13}}\left(\frac{\left|A_2(0)\right|}{\left|A_3(0)\right|}\right)^2$$

Resonant Interactions between Acoustic and Gravity Modes

➢ Numerical results for a representative example of resonant triad containing two acoustic-inertia modes and one gravity-inertia mode:



Determination of a resonant triad involving a long inertio-acoustic, a short acoustic mode and a short gravity mode. The acoustic modes have k = 1 vertical structure, while the gravity mode has a k = 2 vertical structure.

➢ Numerical results for a representative example of resonant triad containing two acoustic-inertia modes and one gravity-inertia mode:

Mode 1: **unstable (pump) mode**

Acoustic mode with k = 1; s = 476, l = 0 (first symmetric mode)

Mode 2:

Acoustic mode with k = 1, s = 1, l = 0 (first symmetric mode)

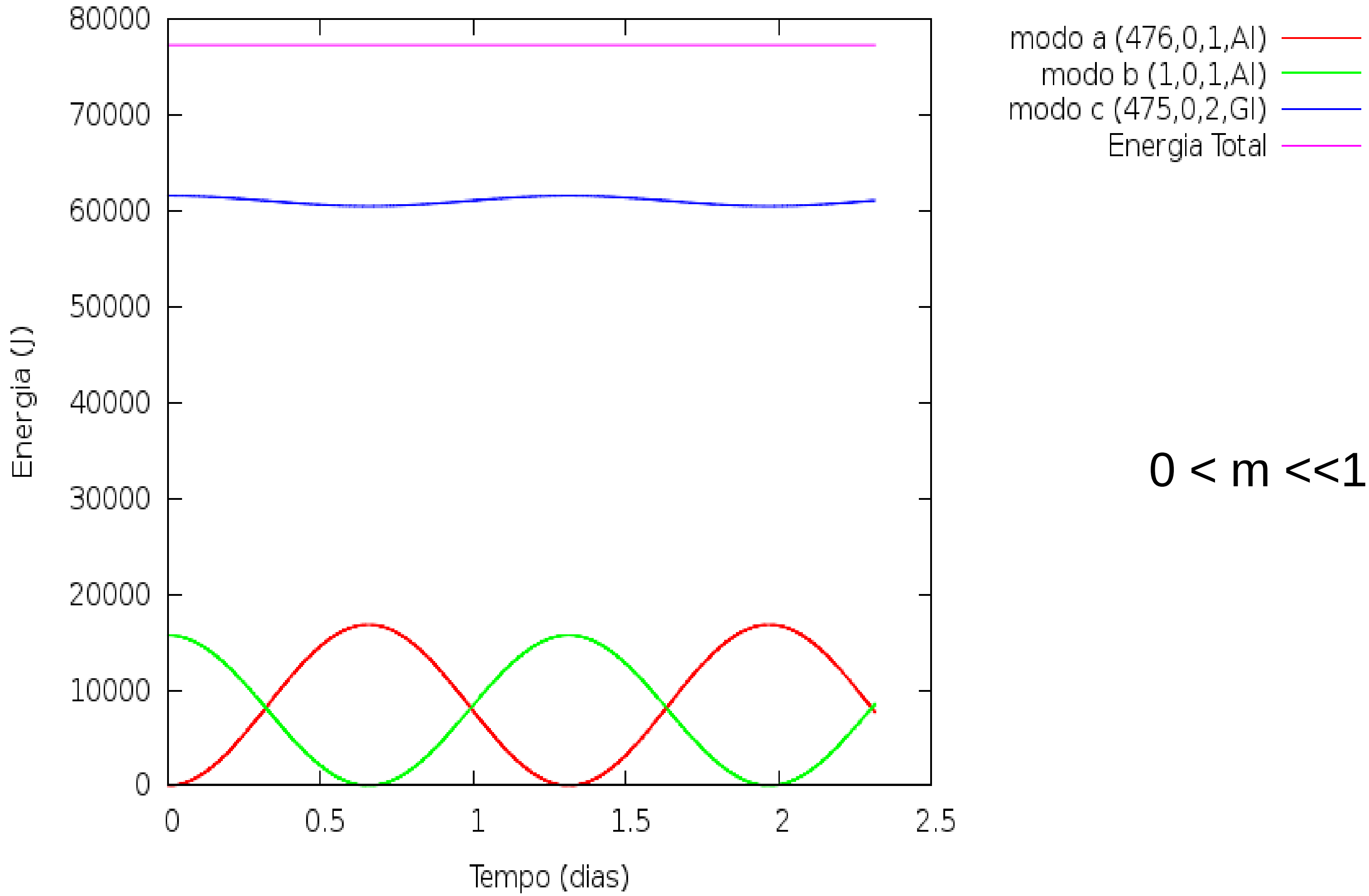
Mode 3:

Gravity mode with k = 2, s = 475, l = 0 (first symmetric mode)

Mode 1	Mode 2	Mode 3	σ_1 (cHz)	σ_2 (cHz)	σ_3 (cHz)	ET_1 (J)	ET_2 (J)	ET_3 (J)	α_1^{23}	α_2^{13}	α_3^{12}
(1,476,0,A)	(1,1,0,A)	(2,475,0, G)	6.293	5.888	0.407	1.3×10^{10}	1.3×10^{20}	7×10^{6}	4×10^{11}	3.7×10^{11}	2.6×10^{10}

Resonant Interactions between Acoustic and Gravity Modes

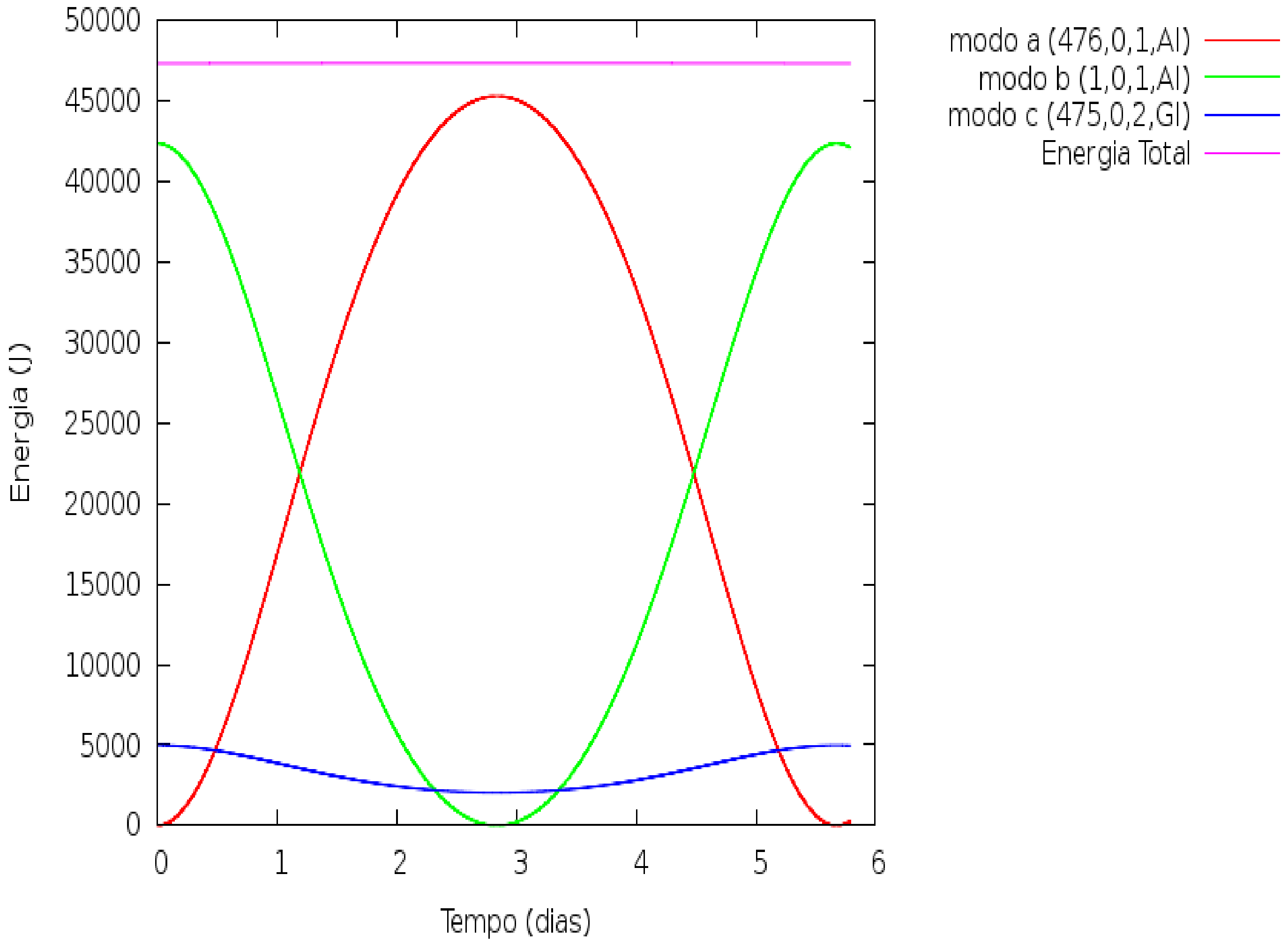
➢ Numerical results for a representative example of resonant triad containing two acoustic-inertia modes and one gravity-inertia mode:



$0 < m << 1$

Resonant Interactions between Acoustic and Gravity Modes

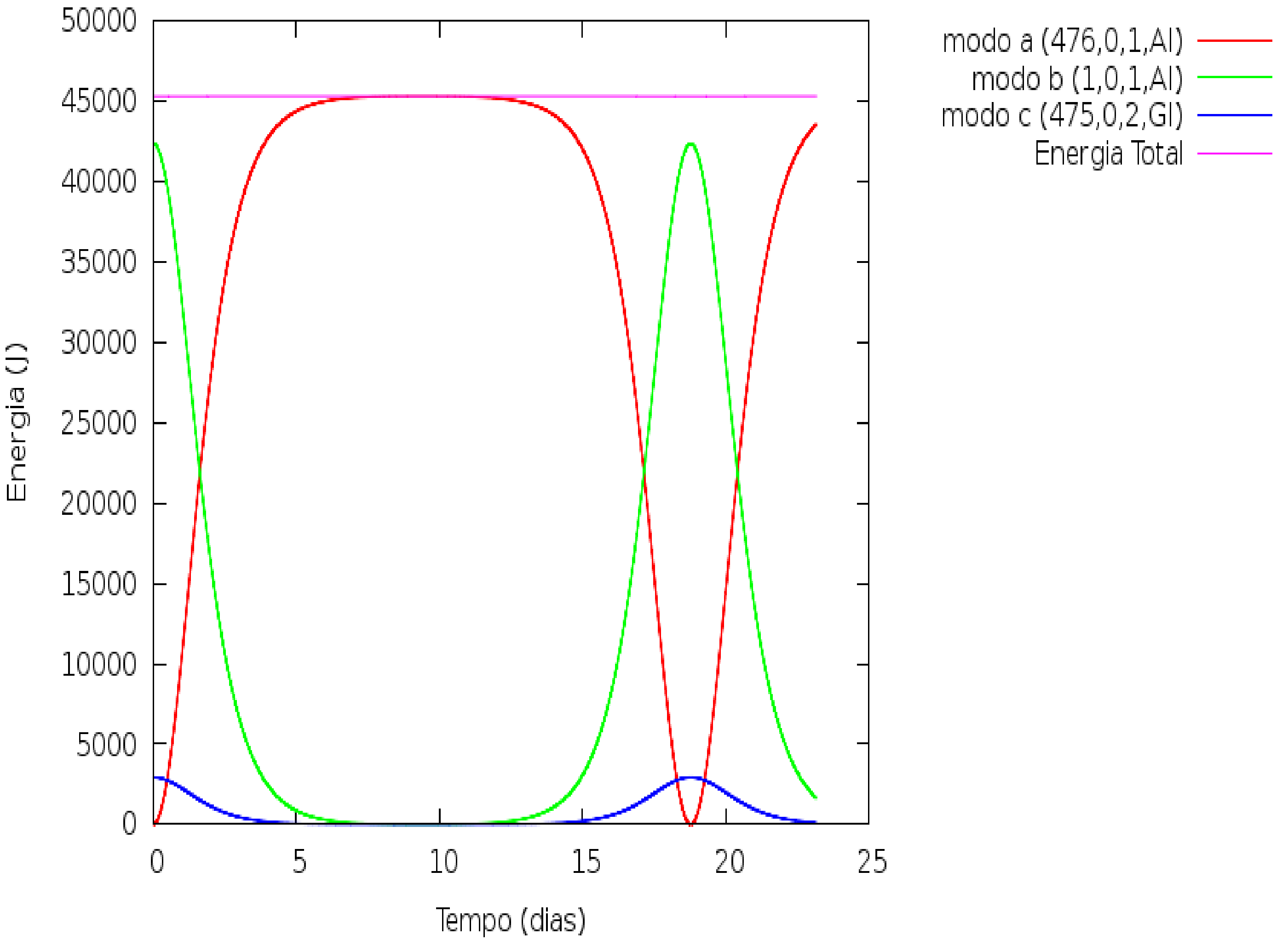
➢Numerical results for a representative example of resonant triad containing two acoustic-inertia modes and one gravity-inertia mode:



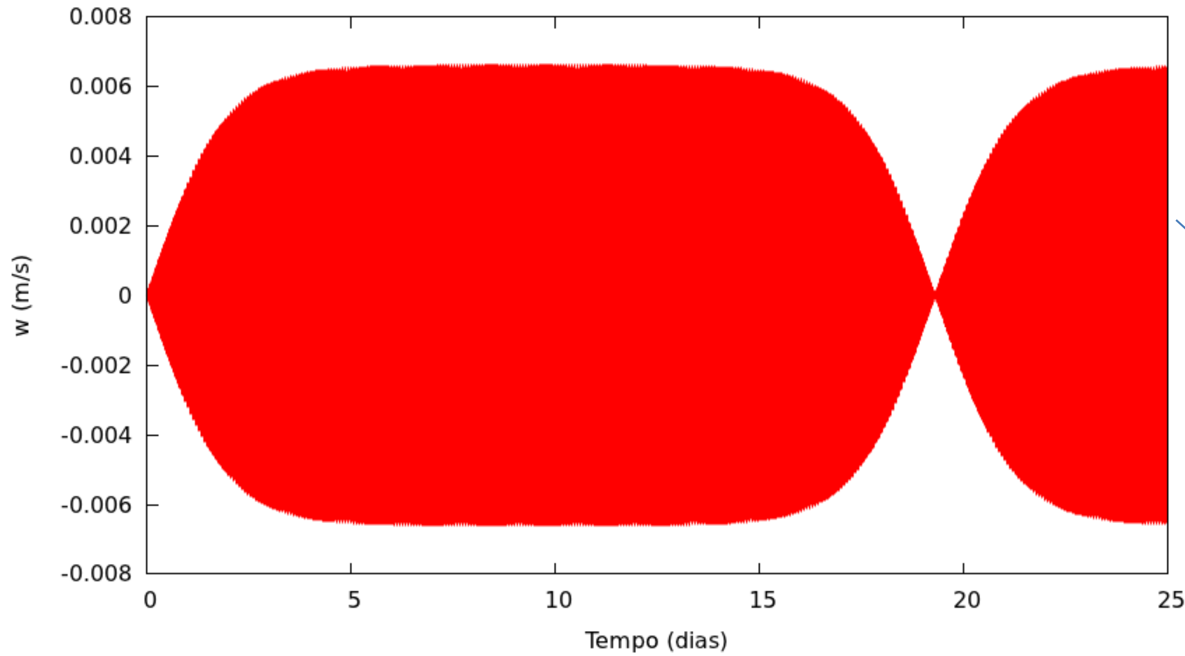
0 < m <1

Resonant Interactions between Acoustic and Gravity Modes

- Numerical results for a representative example of resonant triad containing two acoustic-inertia modes and one gravity-inertia mode:



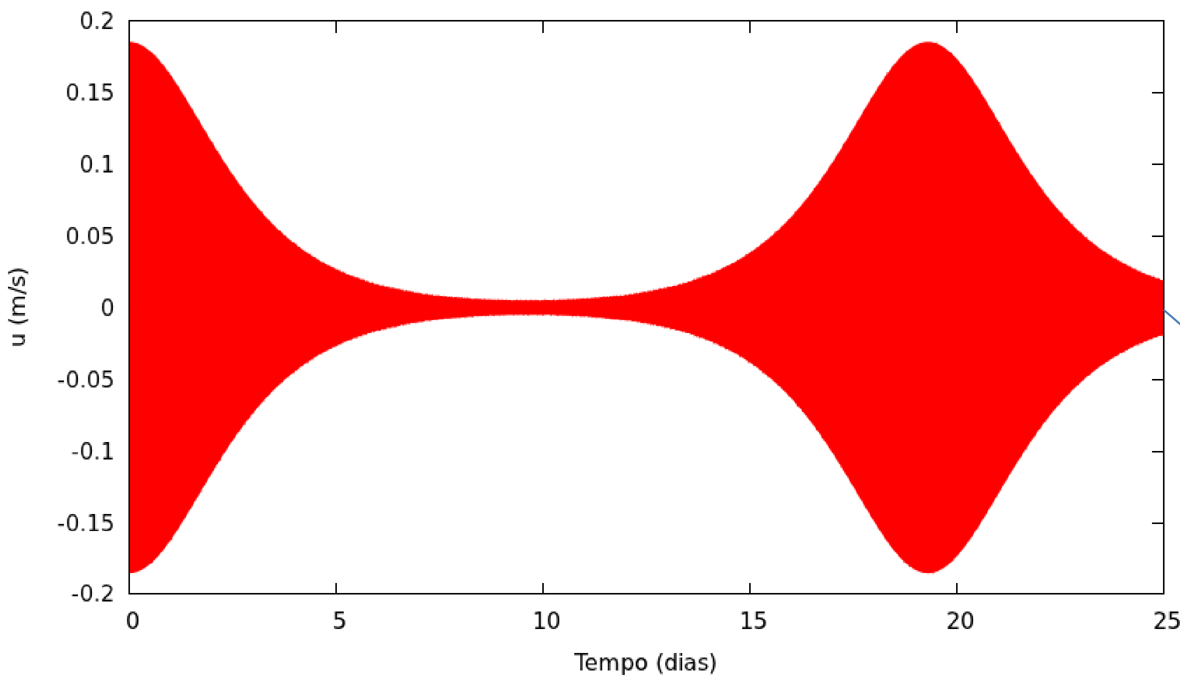
0<< m <1



Vertical velocity at φ = 10ºS and z = 9Km;

Short acoustic mode activity.

$0 << m < 1$



zonal velocity at φ = 0º and z = 4.5Km

Short gravity mode activity.

Summary and Remarks

- Here we have investigated the possibility of resonant interactions involving inertio-acoustic and inertio-gravity modes in a shallow-nonhydrostatic global atmospheric model (weakly nonlinear extension of Kasahara and Qian (2000))

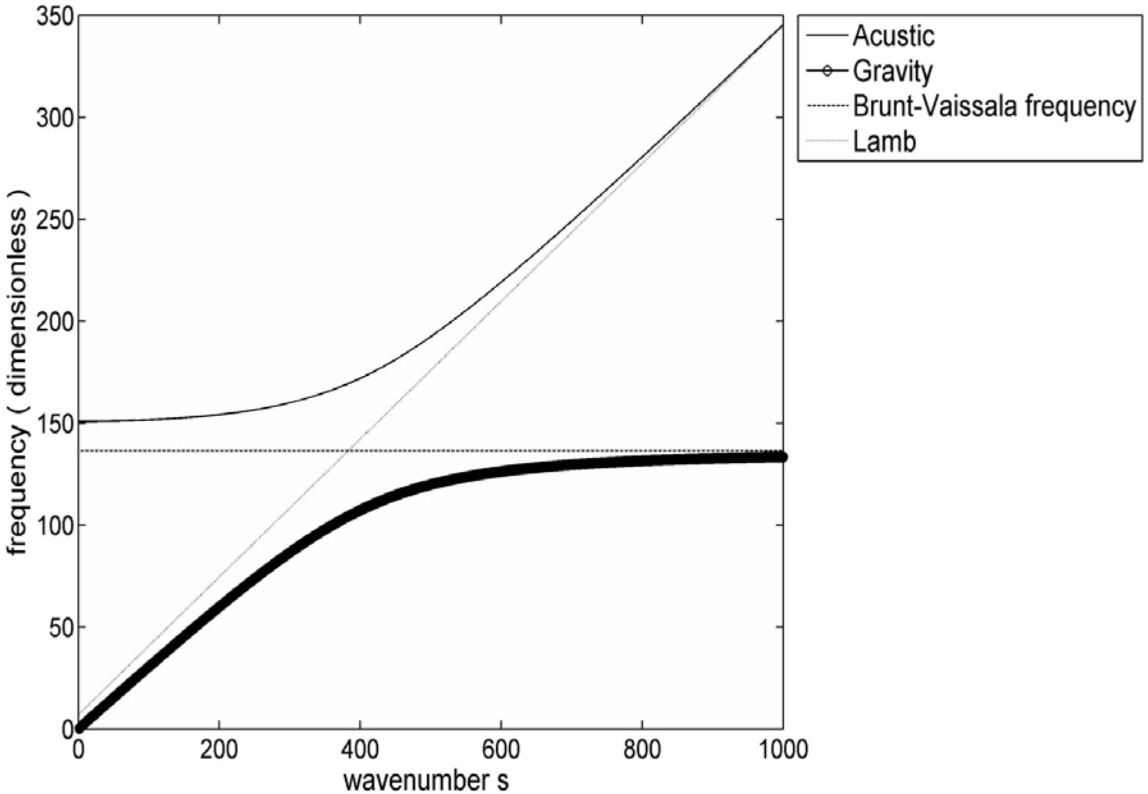
- For the internal modes (rigid lid boundary condition), we found that the only possibility for such resonances is that one gravity mode interacts with two acoustic modes (similar to Rossby-gravity-gravity interaction in the hydrostatic dynamics);

- This kind of resonant interaction can potentially yield vacillations in the dynamical fields with periods varying from a daily (and intra-diurnal) time-scale up to almost a month long, depending on the way in which the initial energy is distributed on the triad components;

- Acoustic modes are usually filtered out from numerical models to avoid computational constraints associated with explicit numerical schemes, **even in nonhydrostatic models**;

Next Steps of the Project

➢ To investigate the possibility of resonant interactions for the limiting case of vertical modes where $z_T \rightarrow \infty$.



➢ To study the possibility of long-short wave interactions.

➢ To investigate the dynamics of these resonant interactions with the inclusion of diabatic effects;